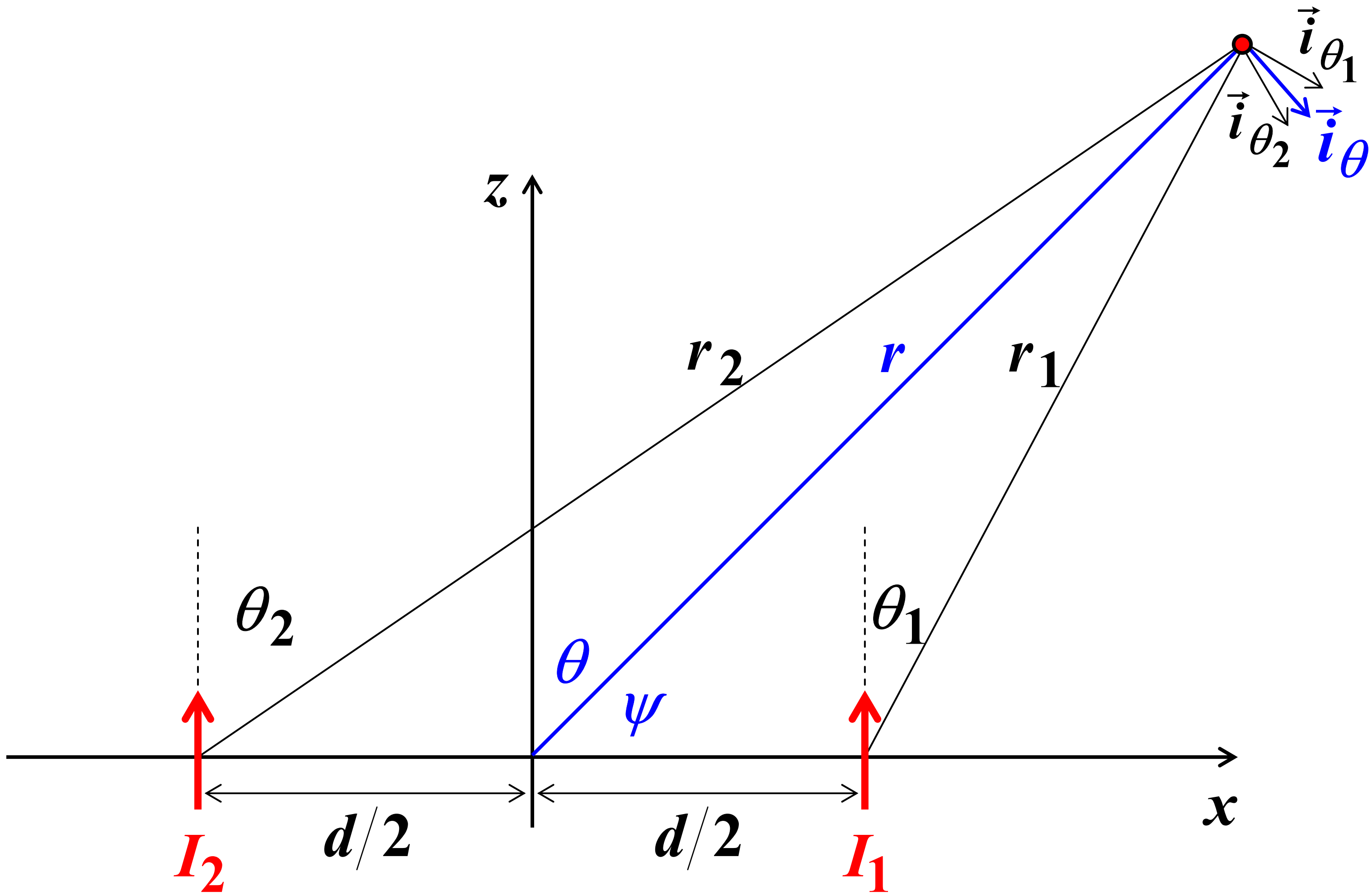
# Simple array of two antennas

**Consider two identical parallel Hertzian dipoles separated by a distance $d$**



**The dipole currents have the same amplitude and total phase difference** α

$$I_1(t) = I_o \cos(\omega t + \alpha/2) \underbrace{\Rightarrow}_{phasor} \mathrm{I}_1 = I_o\, e^{j\alpha/2}$$

$$I_2(t) = I_o \cos(\omega t - \alpha/2) \underbrace{\Rightarrow}_{phasor} \mathrm{I}_2 = I_o\, e^{-j\alpha/2}$$

**The electric far–field components at the observation point are**

$$\vec{\mathrm{E}}_1 \approx \vec{i}_{\theta_1} \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta I_o \Delta z\, e^{-j\beta r_1 + j\alpha/2}}{4\pi r_1} \sin\theta_1$$

$$\vec{\mathrm{E}}_2 \approx \vec{i}_{\theta_2} \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta I_o \Delta z\, e^{-j\beta r_2 - j\alpha/2}}{4\pi r_2} \sin\theta_2$$

**At long distance, we have**

$$r >> d$$

$$\theta_1 \approx \theta_2 \approx \theta$$

$$\vec{i}_{\theta 1} \approx \vec{i}_{\theta 2} \approx \vec{i}_{\theta}$$

$$r_1 \approx r - \frac{d}{2}\cos\psi \qquad\qquad r_2 \approx r + \frac{d}{2}\cos\psi$$

**and the field components can be written as**

$$\vec{E}_1 \approx \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta I_o \Delta z \, e^{-j\beta(r-(d/2)\cos\psi)+j\alpha/2}}{4\pi\left(r - \cancel{(d/2)\cos\psi}\right)} \sin\theta$$

$$\vec{E}_2 \approx \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta I_o \Delta z \, e^{-j\beta(r+(d/2)\cos\psi)-j\alpha/2}}{4\pi\left(r + \cancel{(d/2)\cos\psi}\right)} \sin\theta$$

**After applying the approximations, the two components can be combined to give the total electric field**

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 \approx \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta I_o \Delta z \sin\theta}{4\pi r} e^{-j\beta r} \times \left( e^{j\left(\beta (d/2)\cos\psi + \alpha/2\right)} + e^{-j\left(\beta (d/2)\cos\psi + \alpha/2\right)} \right)$$

**The final result is**

$$\vec{\mathbf{E}} \approx \underbrace{\vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta I_o \Delta z \sin\theta}{4\pi r} e^{-j\beta r}}_{\text{field of a Hertzian dipole located at the center of the array}} \; \underbrace{2\cos\left(\frac{\beta d \cos\psi + \alpha}{2}\right)}_{\text{array factor}}$$

**The resultant radiation pattern of the electric field is proportional to**

$$\underbrace{\sin\theta}_{\text{unit pattern}} \quad \times \quad \underbrace{\cos\left(\frac{\beta\, d \cos\psi + \alpha}{2}\right)}_{\text{group pattern}}$$

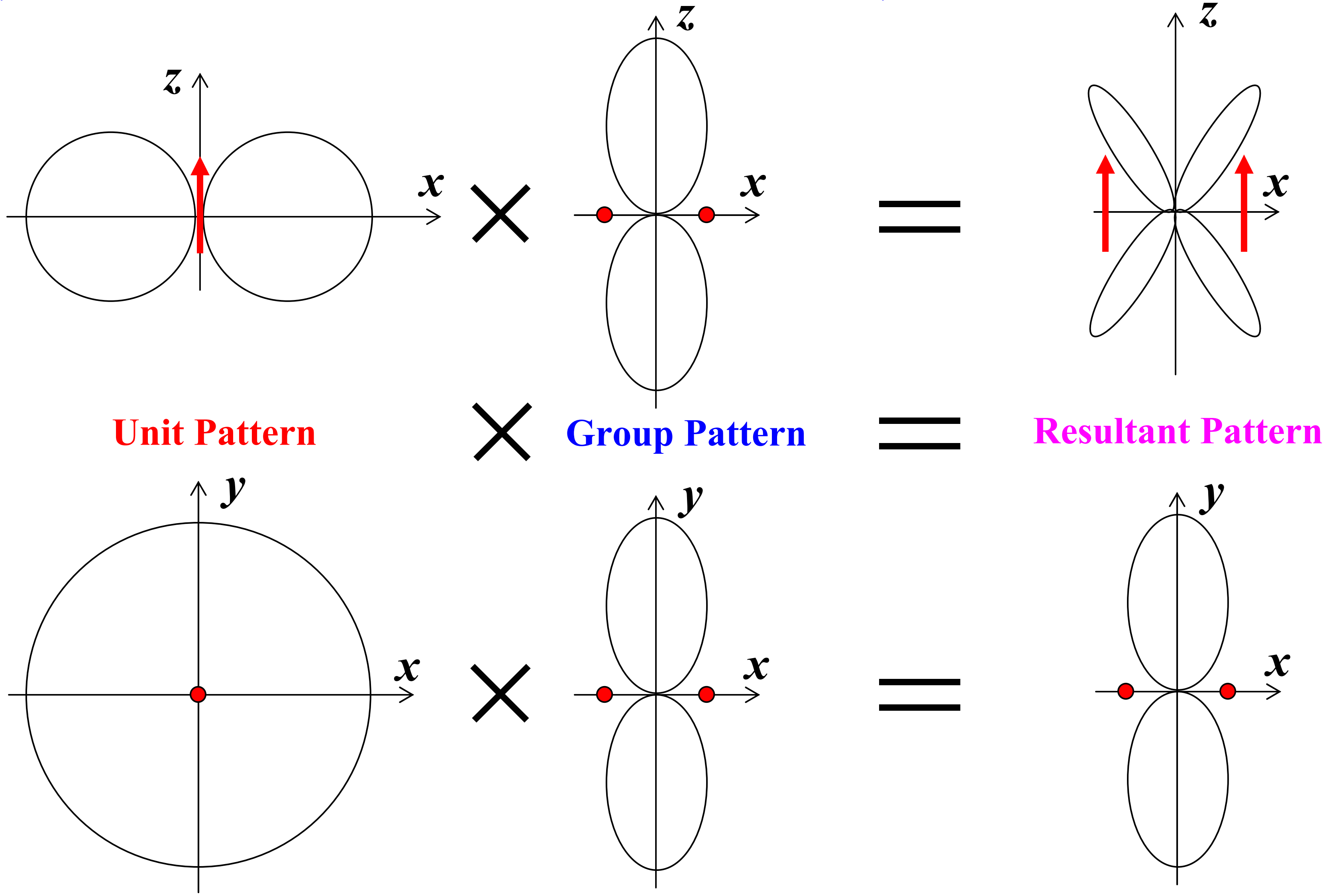
**The unit pattern is proportional to the radiation pattern of the individual antennas, assumed to be identical.**

**The group pattern is proportional to the radiation pattern the array would have with isotropic antennas.**

- **Note: on the $x$–$y$ plane, $\psi$ coincides with the azimuthal angle $\varphi$.**

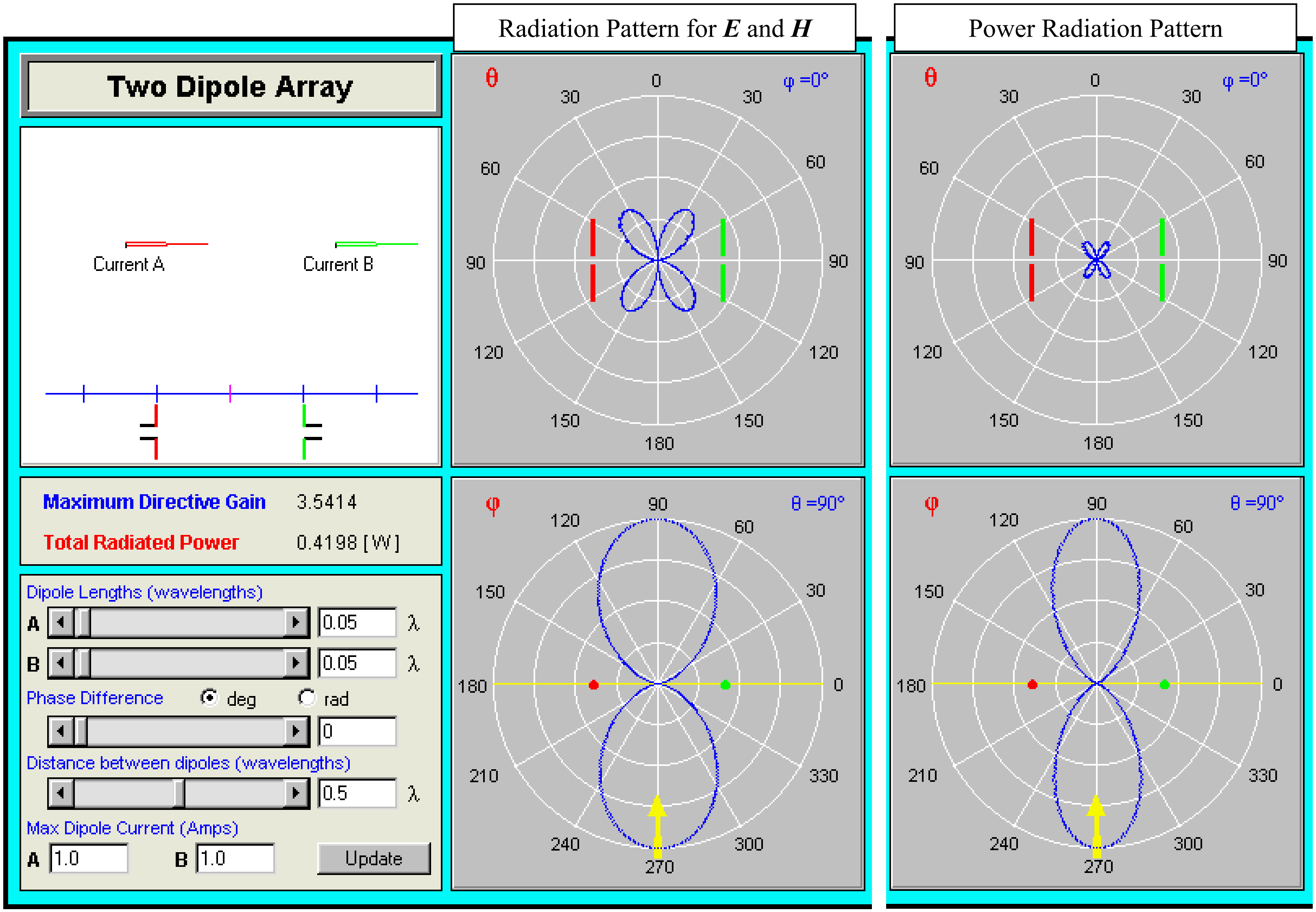
**Following are examples of two–antenna arrays with specific values of dipole distance and current phase difference.**

$d = \lambda / 2$  $\alpha = 0$  **Broad-side pattern**

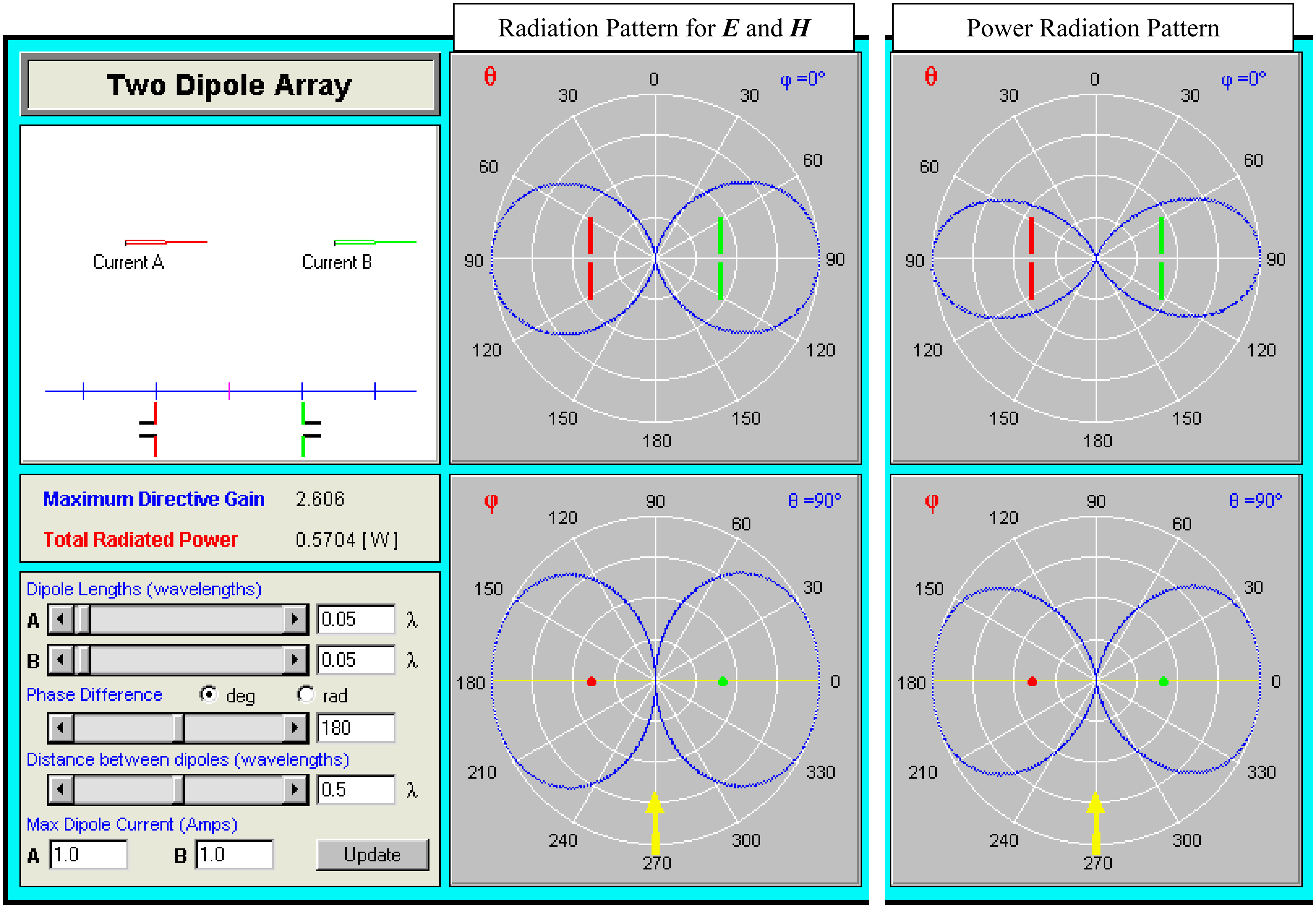


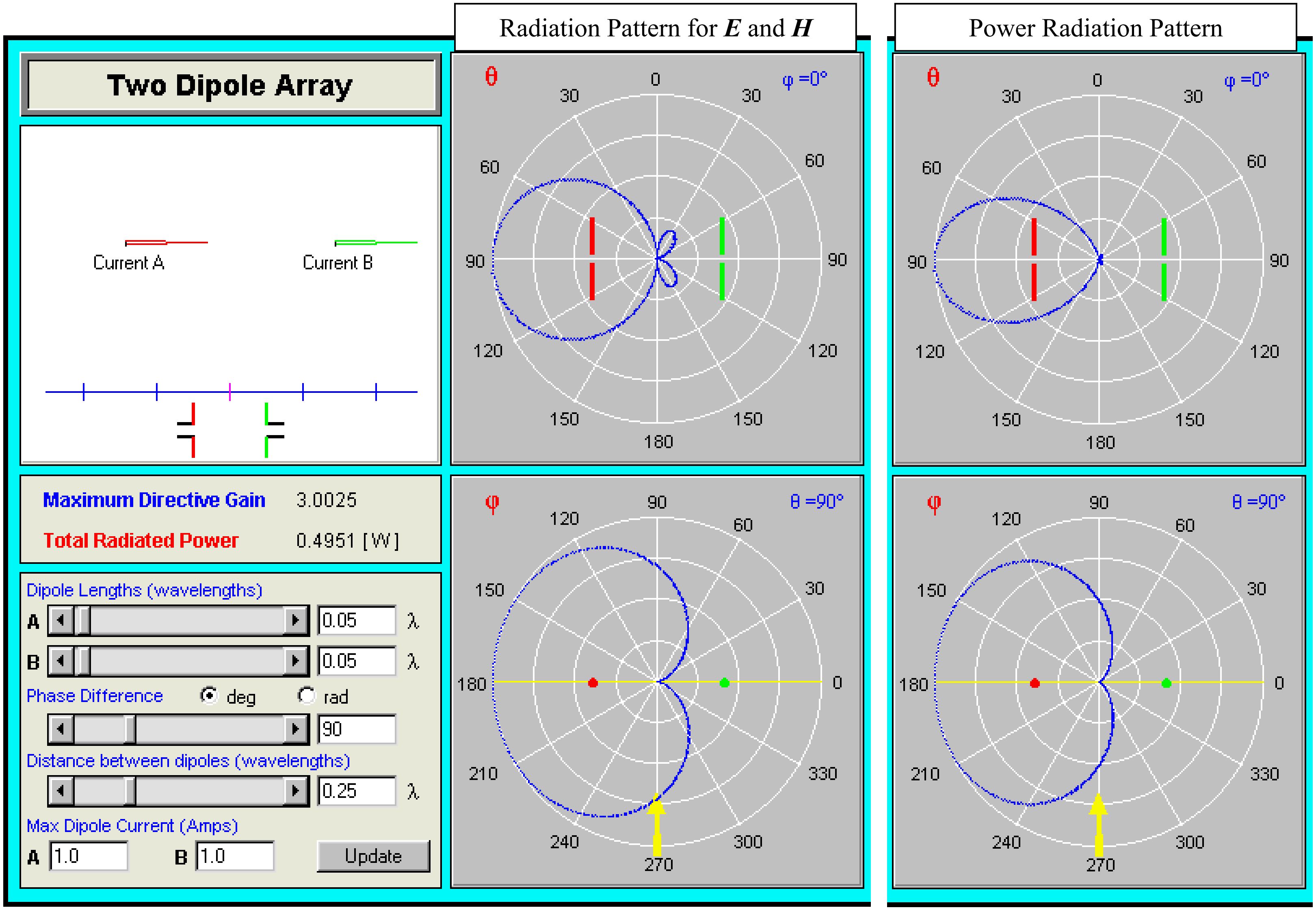
$d = \lambda / 2 \qquad \alpha = 0$ **Broad-side pattern**



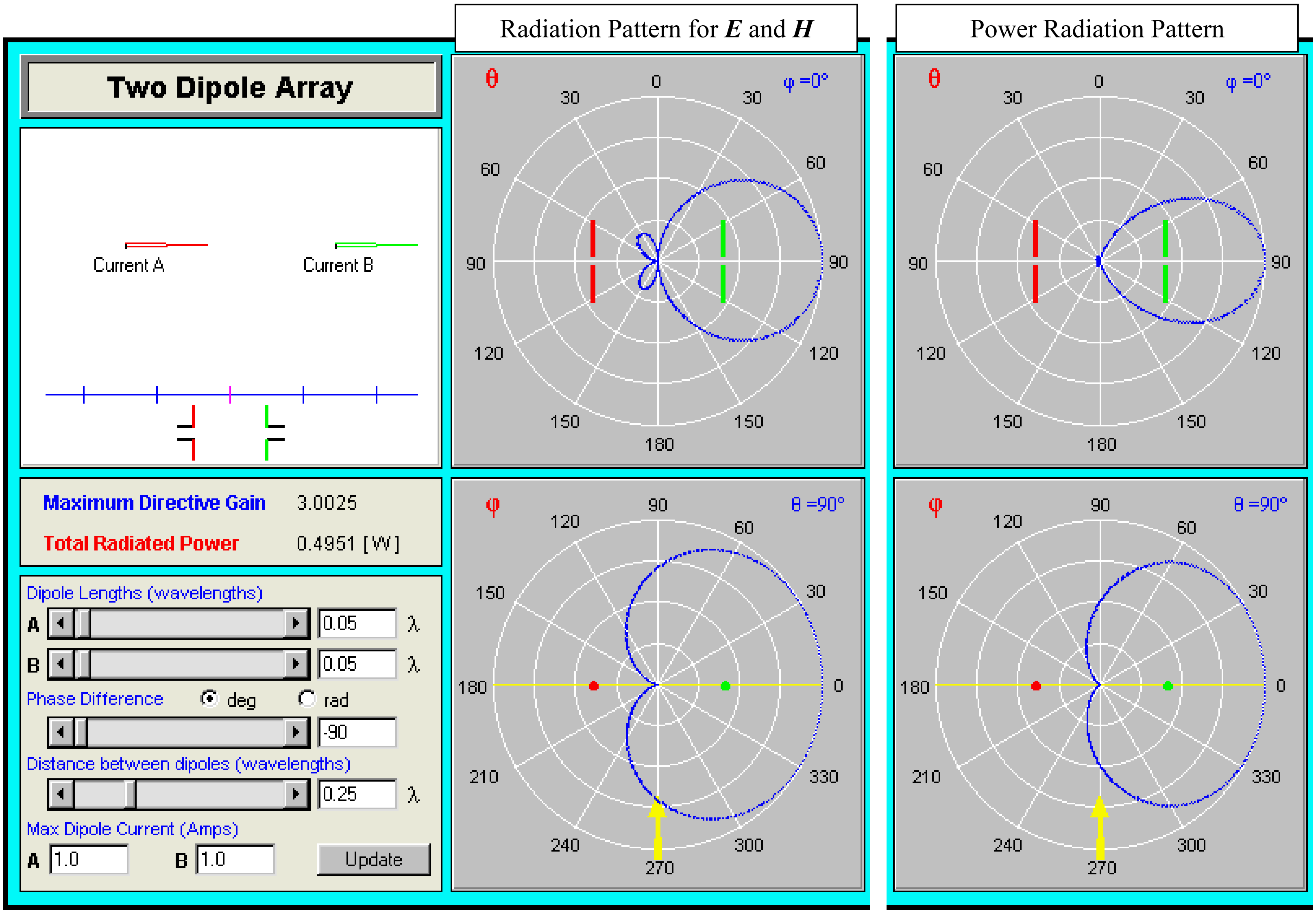
$d = \lambda / 2 \qquad \alpha = 180^\circ$ **End-fire pattern**



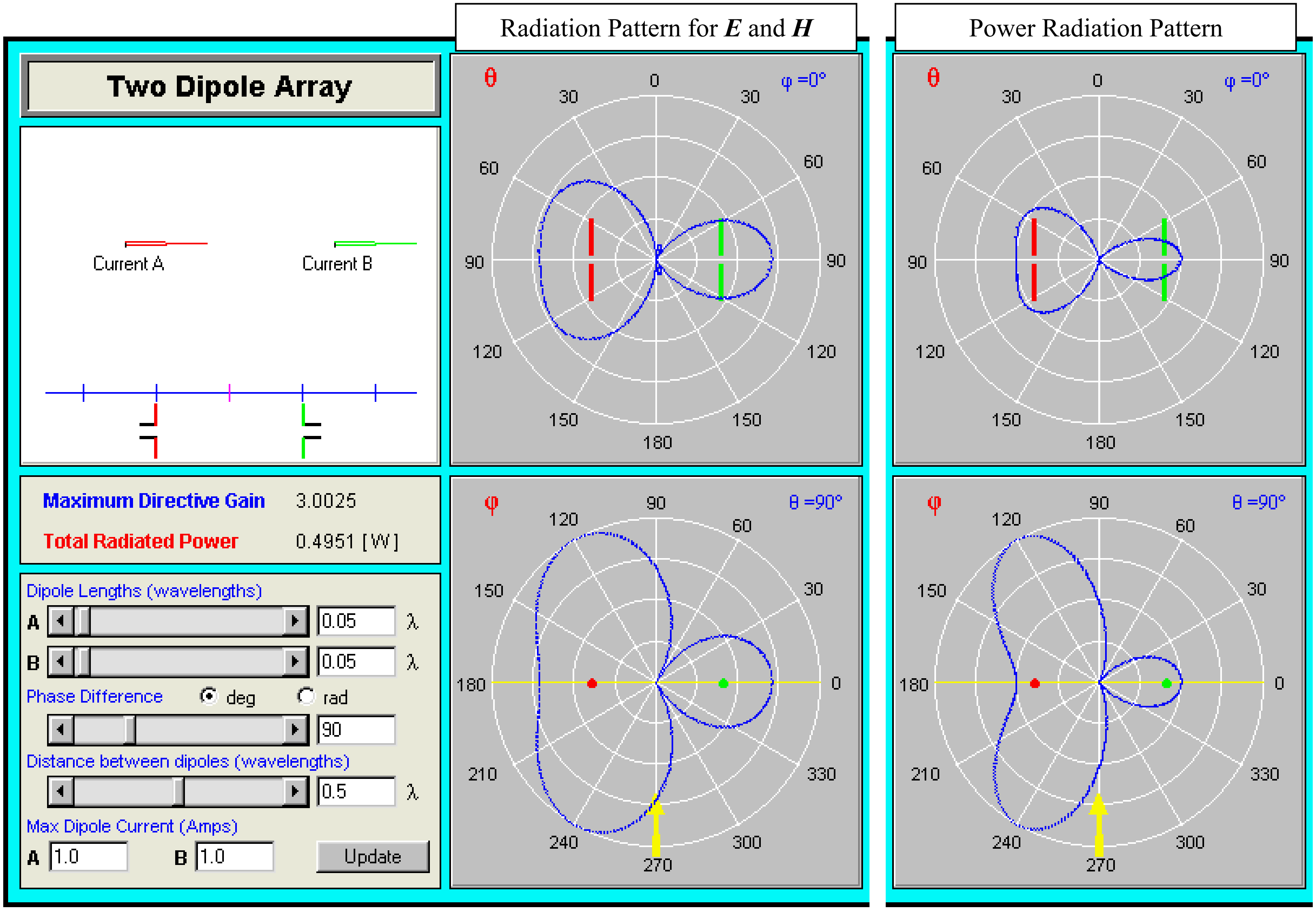
$d = \lambda / 4 \qquad \alpha = 90^\circ$ **Cardioid pattern**



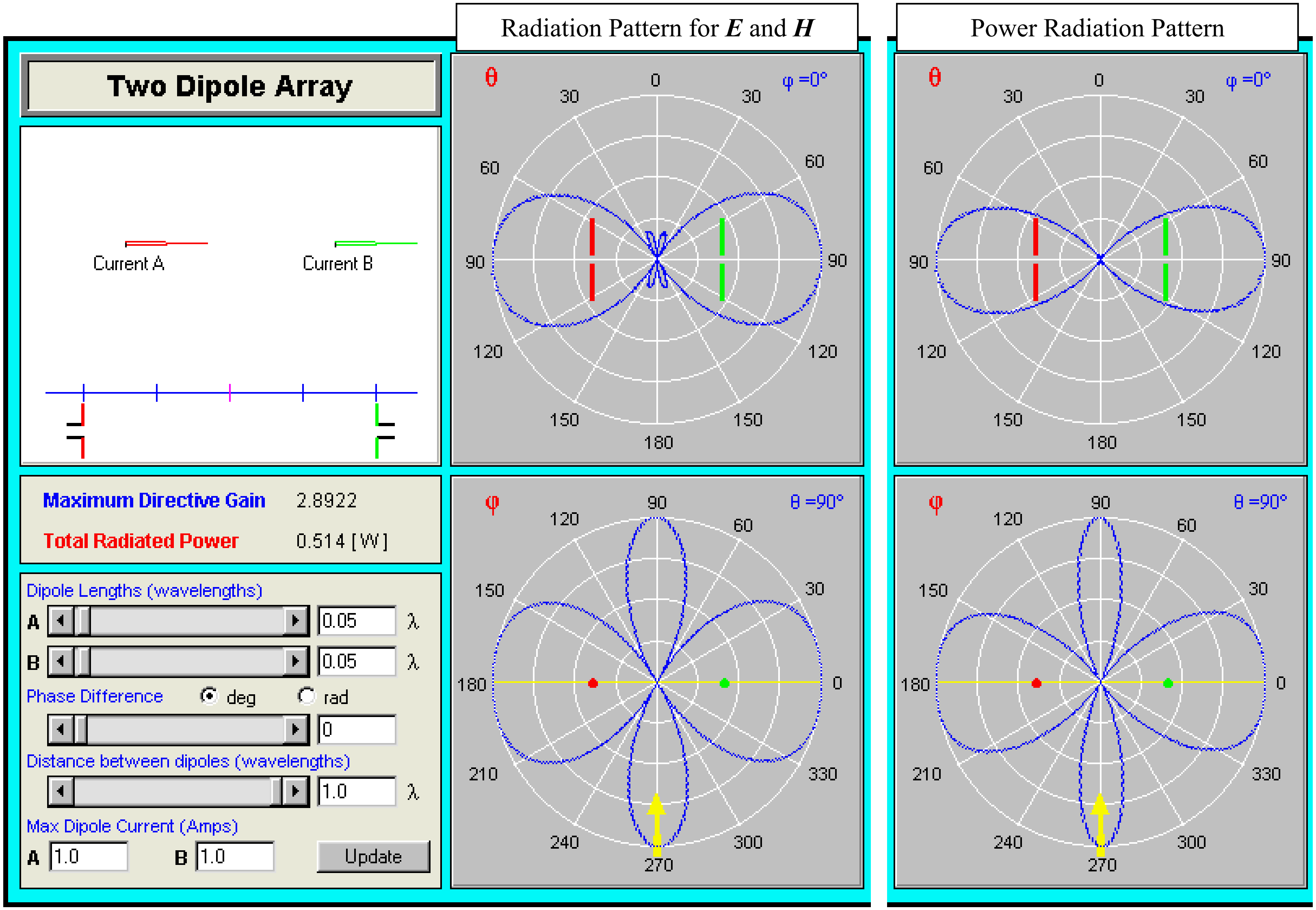
$d = \lambda / 4$ $\alpha = -90^\circ$ **Cardioid pattern**



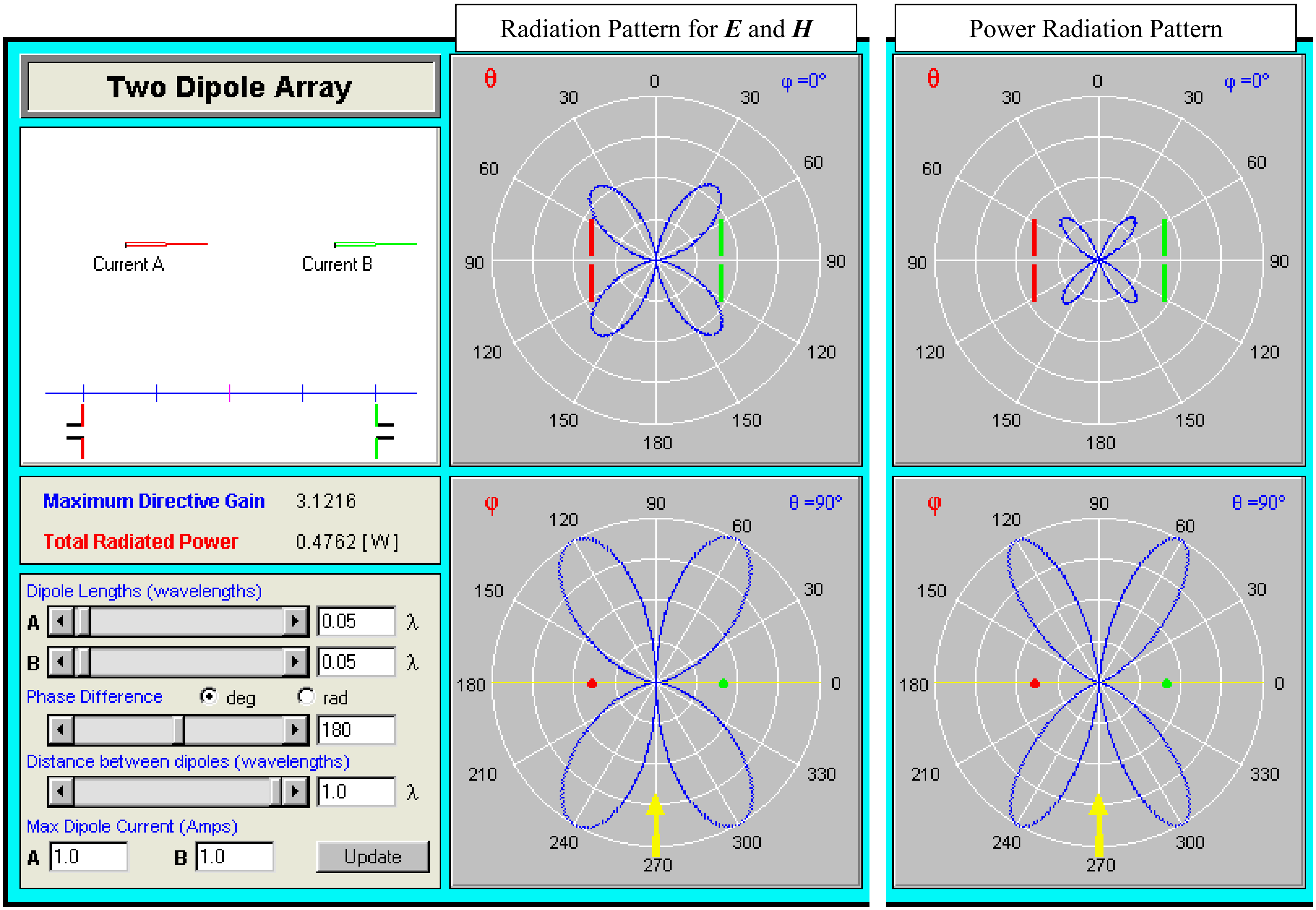
$d = \lambda / 2 \qquad \alpha = 90^\circ$



$d = \lambda \qquad \alpha = 0^\circ$



$$d = \lambda \qquad \alpha = 180^\circ$$



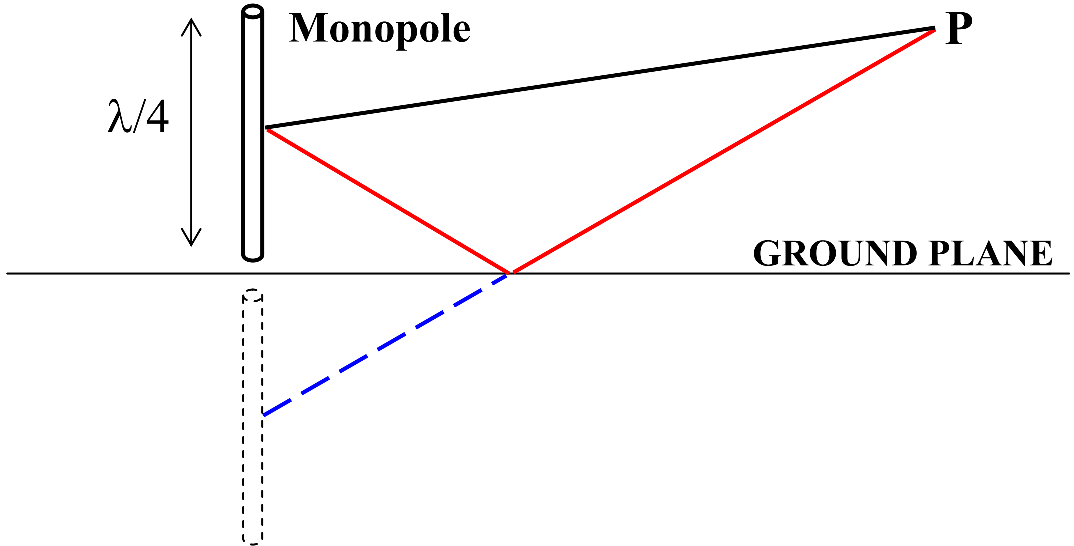
**CASE STUDY - 1) A radio broadcast transmitter is located 15 km West of the city it needs to serve. The FCC standard is to have 25 mV/m electric field strength in the city. How much radiation power must be provided to a quarter wavelength monopole?**

**We consider θ=90° for transmission in the plane perpendicular to the antenna**

$$\vec{\mathbf{E}} = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\, e^{-j\beta r}}{2\pi r} \frac{I_{max}}{\sin 90°} \cos\left(\frac{\pi \cos 90°}{2}\right) \qquad \lambda/2 \text{ dipole}$$

$$\left|\vec{\mathbf{E}}\right| = 120\pi \frac{I_{max}}{2\pi \times 15{,}000} = 0.025 \text{ V/m} \quad \Rightarrow \quad I_{max} = 6.25 \text{ A}$$

**In a monopole, the lower wire is substituted by the ground. The equivalent radiation resistance is half that of the corresponding dipole. Therefore, the total radiated power is half the power radiated by the half-wavelength dipole, for the same current.**




**A perfect ground would act like a metal surface, reflecting 100% of the signal. The ground creates an image of the "missing" wire delivering to a given point above the ground the same signal as a complete dipole. The transmission line connected to the antenna sees only half of the radiation resistance, with total radiated power:**

$$P_{tot} = \frac{1}{2} I_{\max}^2 \underbrace{\sqrt{\frac{\mu}{\varepsilon}} \cdot 0.193978 / 2}_{R_{eq}} = 0.5 \cdot 6.25^2 \cdot \frac{73.07}{2} = 713.6\text{W}$$

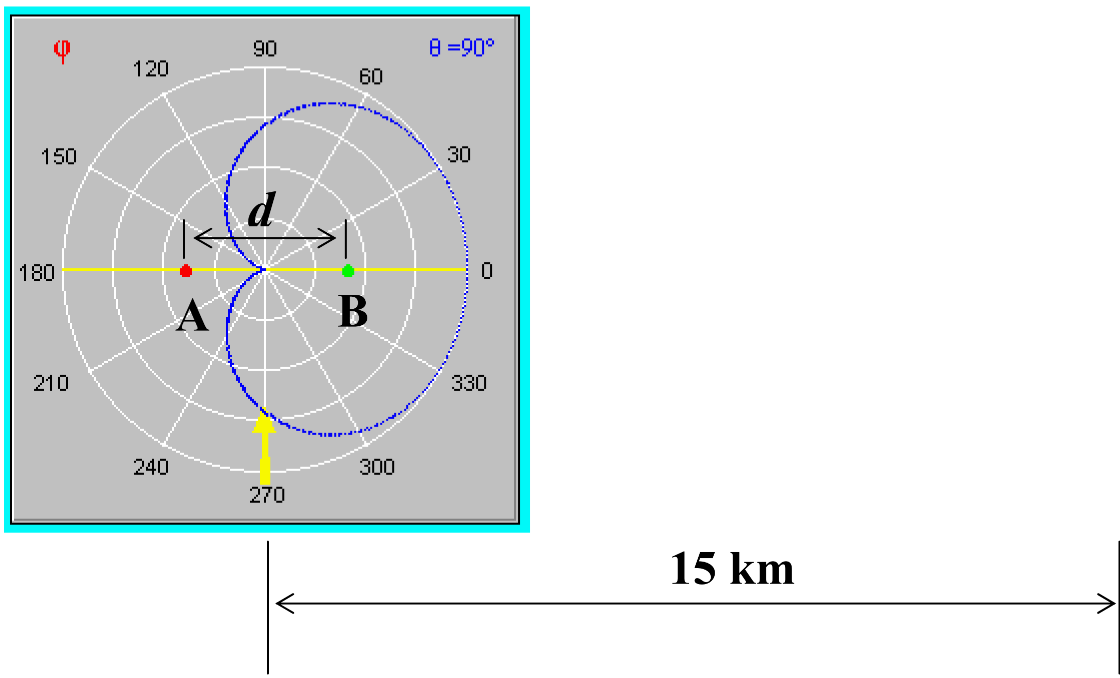
## 2) Improve the design by using a two–antenna array.

**A good choice of array parameters is**

$$d = \lambda / 4$$

$$\text{phase(antenna B)} - \text{phase(antenna A)} = \alpha = -90^\circ$$

**which gives a cardioid pattern**



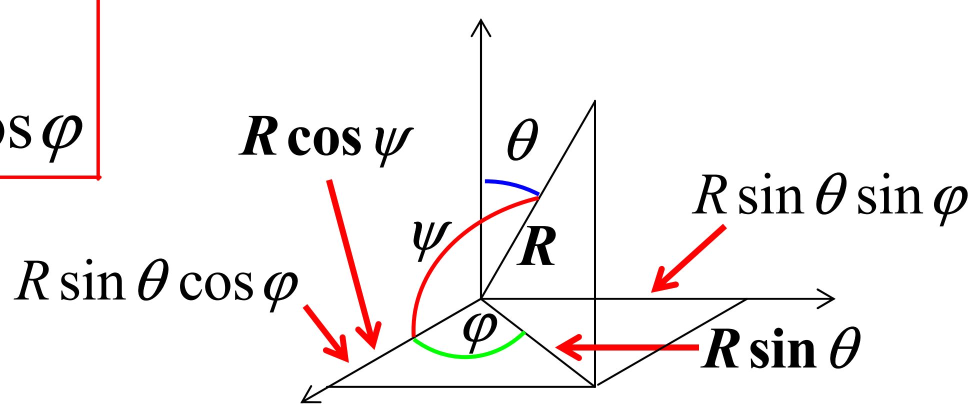
## The Poynting vector is given by

$$\langle \vec{P}(t) \rangle = \underbrace{\vec{i}_r \sqrt{\frac{\mu}{\varepsilon}} \frac{I_{\max}^2}{8\pi^2 r^2 \sin^2\theta} \cos^2\left(\frac{\pi\cos\theta}{2}\right)}_{\lambda/2 \text{ dipole Poynting vector}} \underbrace{4\cos^2\left(\frac{\beta d \cos\psi + \alpha}{2}\right)}_{(\text{array factor})^2}$$

$$= \vec{i}_r \sqrt{\frac{\mu}{\varepsilon}} \frac{I_{\max}^2}{8\pi^2 r^2 \sin^2\theta} \cos^2\left(\frac{\pi\cos\theta}{2}\right) \underbrace{4\cos^2\left(\frac{\pi}{4}\cos\psi - \frac{\pi}{4}\right)}_{(\text{array factor})^2}$$

**Note**

$$\cos\psi = \sin\theta\cos\varphi$$



**The total radiated power is**

**Only π/2 because it is a monopole**

$$P_{tot} = r^2 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} \left\langle \vec{P}(t) \right\rangle d\varphi$$

$$P_{tot} = r^2 \int_0^{\pi/2} \sqrt{\frac{\mu}{\varepsilon}} \frac{I_{\max}^2}{8\pi^2 r^2 \sin^2\theta} \cos^2\left(\frac{\pi\cos\theta}{2}\right) \sin\theta \, d\theta$$

$$\int_0^{2\pi} \underbrace{4\cos^2\left(\frac{\pi}{4}\sin\theta\cos\varphi - \frac{\pi}{4}\right)}_{(\text{array factor})^2} d\varphi$$

**The integral over the azimuthal angle φ gives**

$$4\int_0^{2\pi} \cos^2\left(\frac{\pi}{4}\sin\theta\cos\varphi - \frac{\pi}{4}\right)\, d\varphi$$

$$= 4\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}\sin\theta\cos\varphi - \frac{\pi}{2}\right)\right) d\varphi$$

$$= 4\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi}{2}\sin\theta\cos\varphi\right)\right) d\varphi = 4\pi$$

**For a monopole, we only have the integral**

$$\int_0^{2\pi} d\varphi = 2\pi$$

**In the direction of maximum**

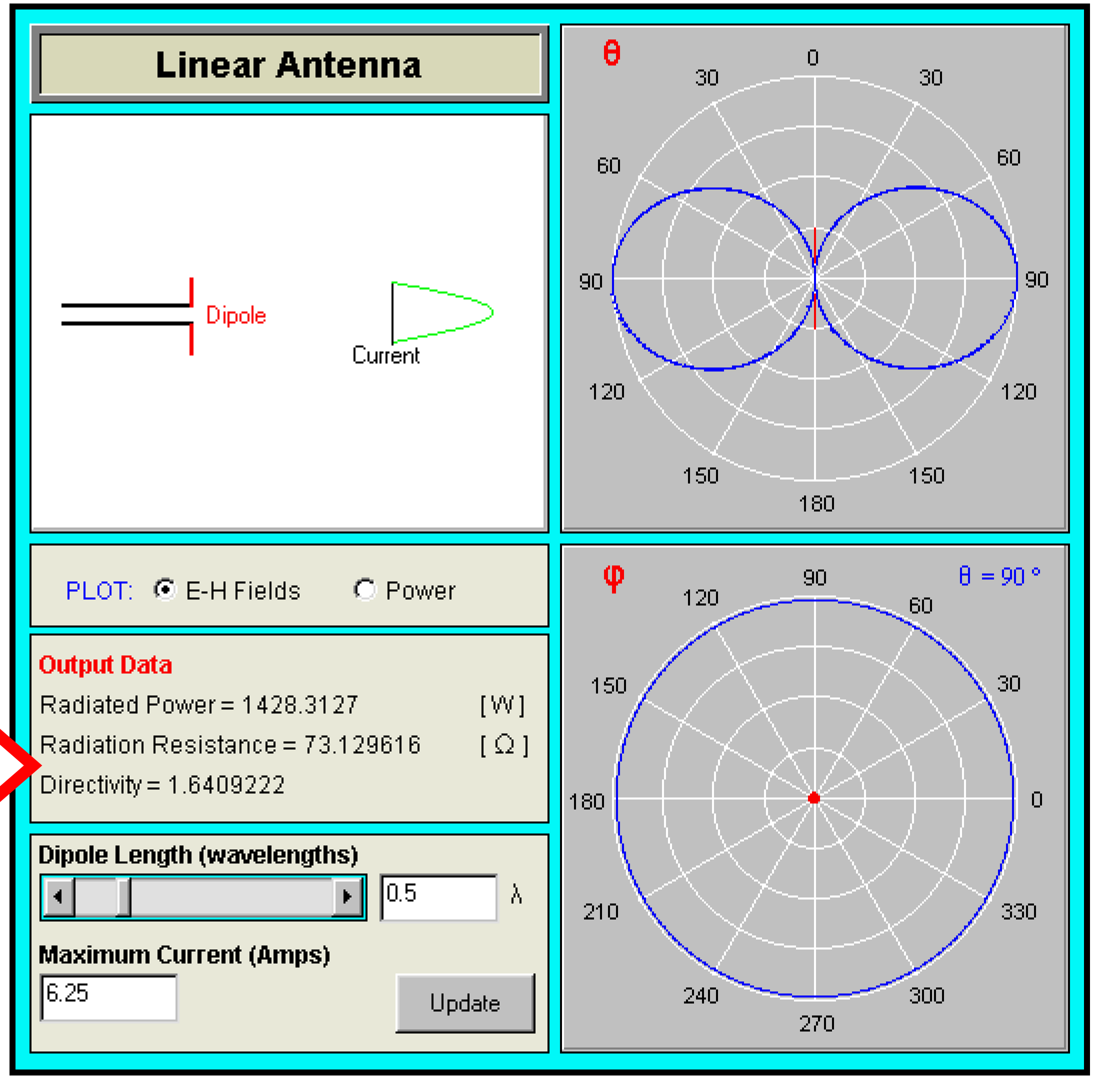
$$\theta = 90° \quad \& \quad \varphi = 90° \Rightarrow \quad \text{array factor} = 2$$

**The same field strength (25 mV/m) is obtained by applying half the current of the original monopole to the array elements (also monopoles)**

$$I_{max} = \frac{6.25}{2} = 3.125 \text{ A}$$

**The total radiated power is proportional to the square of the current, and the integral over φ gives a factor 4π instead of 2π for the array. Overall, the total radiated power needed by the array, to produce the same electric field, is half that of the individual monopole**

$$P_{tot} = \frac{714}{2} = 357 \text{ W}$$

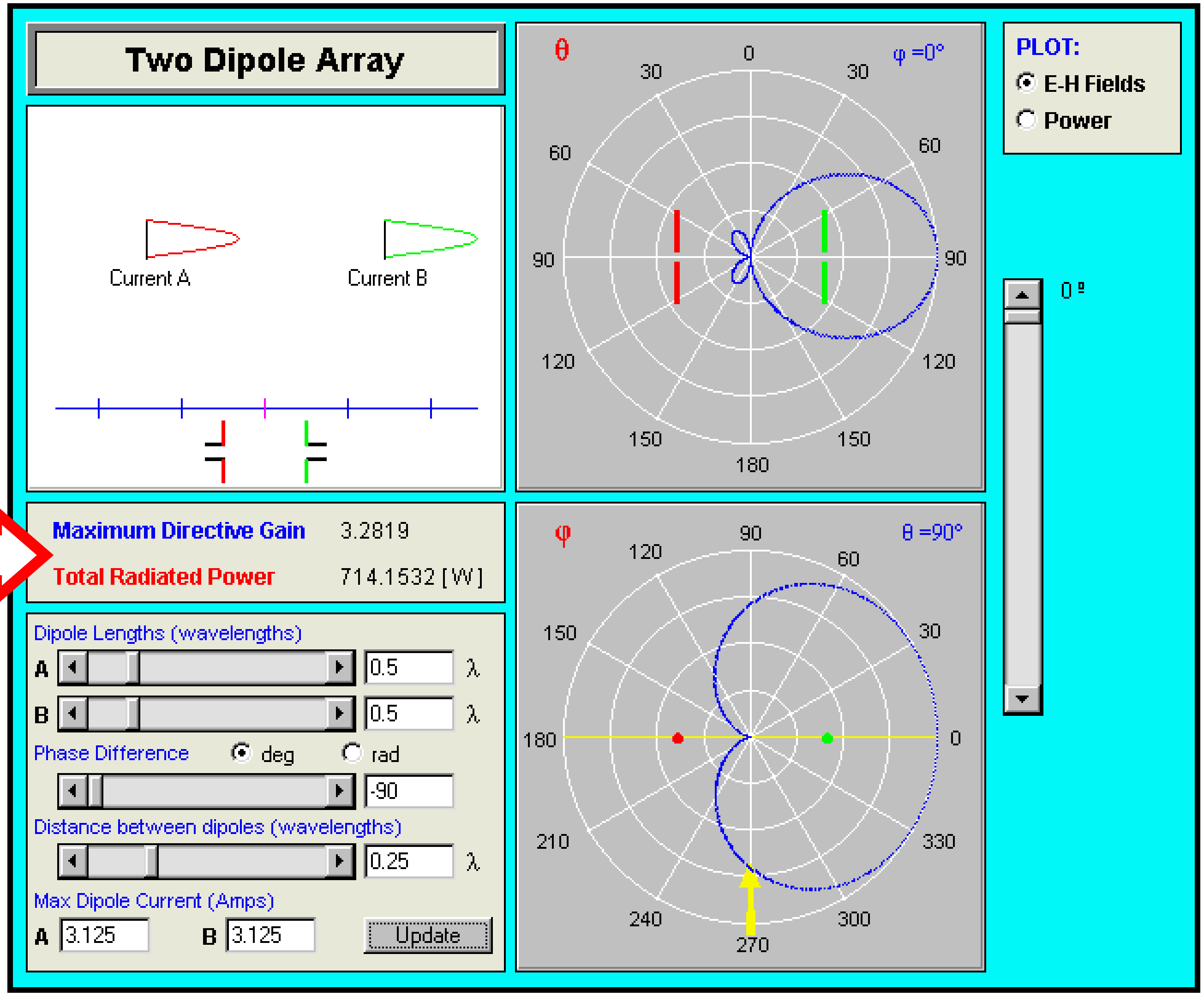



**For a monopole**

**Radiated Power = 0.5×1428.3127**
**= 714.155635 [ W ]**

**Rad. Resistance = 0.5×73.129616**
**= 36.564808 [ Ω ]**

**SAME FEEDING CURRENT FOR BOTH DIPOLE AND MONOPOLE**



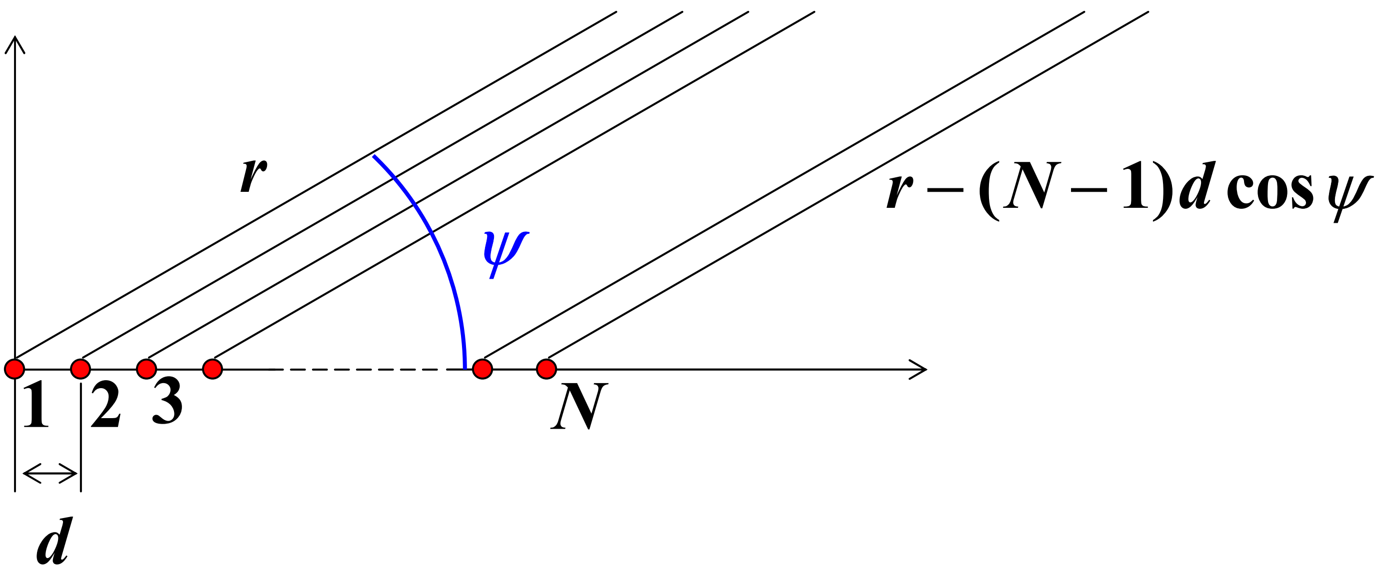

**For monopoles**

**Radiated Power**
**= 0.5×714.1532**
**= 357.0766 [ W ]**

**SAME FEEDING CURRENTS FOR BOTH DIPOLES AND MONOPOLES**

## N-Element Antenna Array

**Assume a uniform array of N identical antennas. The elements are fed by currents with constant amplitude and with phase increasing by an amount α from one to the other. The spacing *d* between the antennas is uniform.**




$$I(1) = I_o; \quad I(2) = I_o e^{j\alpha}; \quad \ldots \quad ; \quad I(N) = I_o e^{j(N-1)\alpha}$$

**The electric field at the observation point $(r,\psi)$ is of the form**

$$\mathbf{E}(r,\psi) = E_o e^{-j\beta r} + E_o e^{-j\beta(r-d\cos\psi)} e^{j\alpha} + \ldots$$

$$\ldots + E_o e^{-j\beta(r-(N-1)d\cos\psi)} e^{j(N-1)\alpha}$$

$$= E_o e^{-j\beta r}\left[1 + e^{j(\beta d\cos\psi+\alpha)} + \ldots + e^{j(N-1)(\beta d\cos\psi+\alpha)}\right]$$

$$= E_o e^{-j\beta r}\frac{1-e^{jN(\beta d\cos\psi+\alpha)}}{1-e^{j(\beta d\cos\psi+\alpha)}}$$

**We have used** $$\sum_{n=0}^{N-1} e^{jn(\beta d\cos\psi+\alpha)} = \frac{1-e^{jN(\beta d\cos\psi+\alpha)}}{1-e^{j(\beta d\cos\psi+\alpha)}}$$

**The magnitude of the electric field is given by**

$$\left|\mathbf{E}(r,\psi)\right| = \left|E_o\right| \left| \frac{1-e^{jN(\beta d\cos\psi+\alpha)}}{1-e^{j(\beta d\cos\psi+\alpha)}} \right|$$

$$= \left|E_o\right| \underbrace{\left| \frac{\sin[N(\beta d\cos\psi+\alpha)/2]}{\sin[(\beta d\cos\psi+\alpha)/2]} \right|}_{\text{array factor}}$$

**We have used**

$$\left|1-e^{jx}\right| = \left|2\ j\sin\frac{x}{2}\ \ e^{jx/2}\right| = 2\left|\sin\frac{x}{2}\right|$$

**We can rewrite**

$$\left|\mathbf{E}(r,\psi)\right| = N\left|E_o\right| \underbrace{\frac{1}{N}\left|\frac{\sin[N(\beta d\cos\psi+\alpha)/2]}{\sin[(\beta d\cos\psi+\alpha)/2]}\right|}_{\text{group pattern}}$$

**The group pattern has**

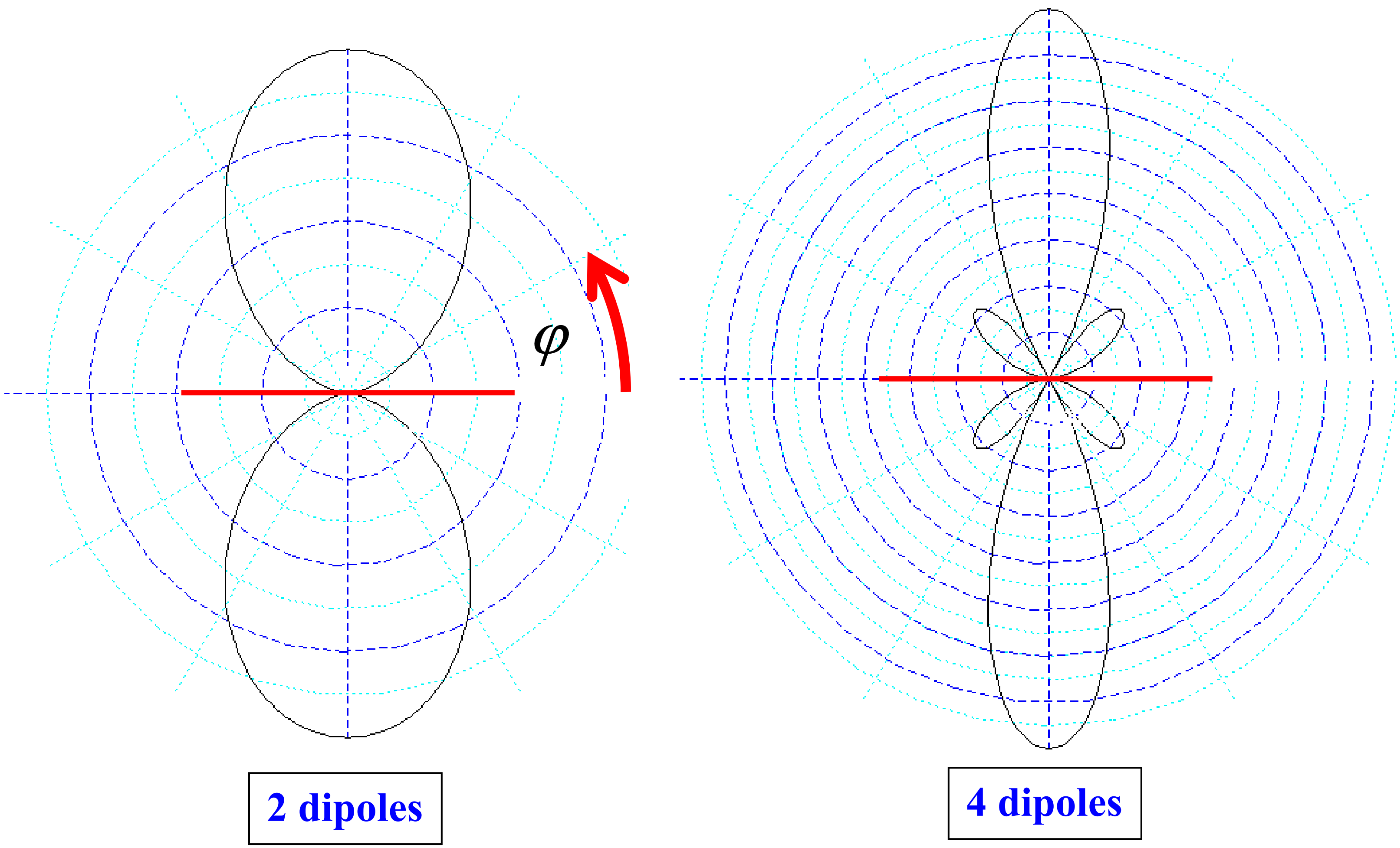
**Maxima when** $\beta d\cos\psi+\alpha = 0,\ 2\pi,\ 4\pi\ldots$

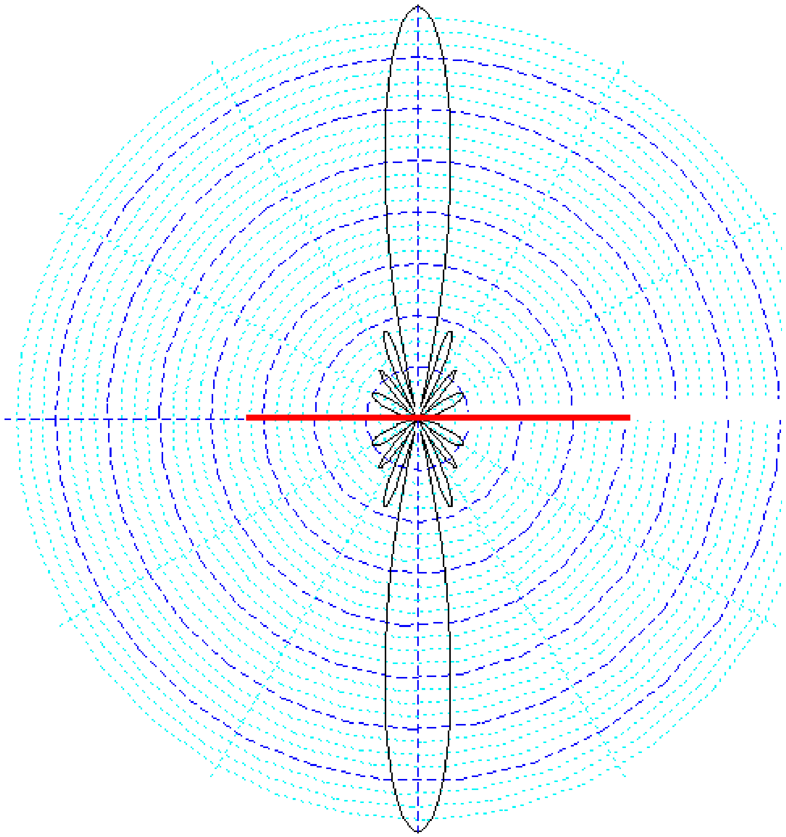
**Nulls when** $N(\beta d\cos\psi+\alpha) = 2m\pi$

**for** $m = \text{integer} \neq 0,\ N,\ 2N,\ldots$

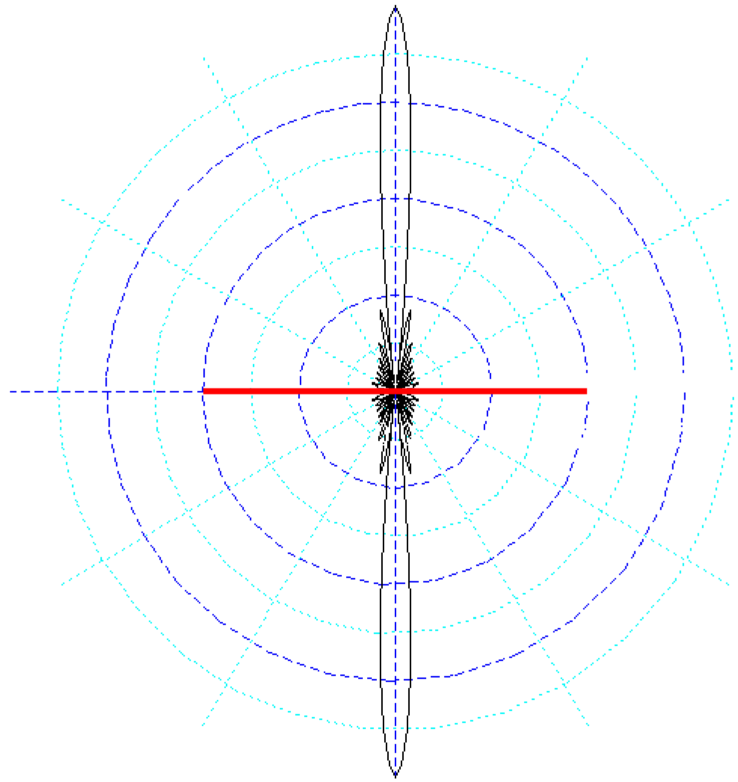
# Examples

## Broadside array (plots on the azimuthal plane, with $\theta = 90°$)



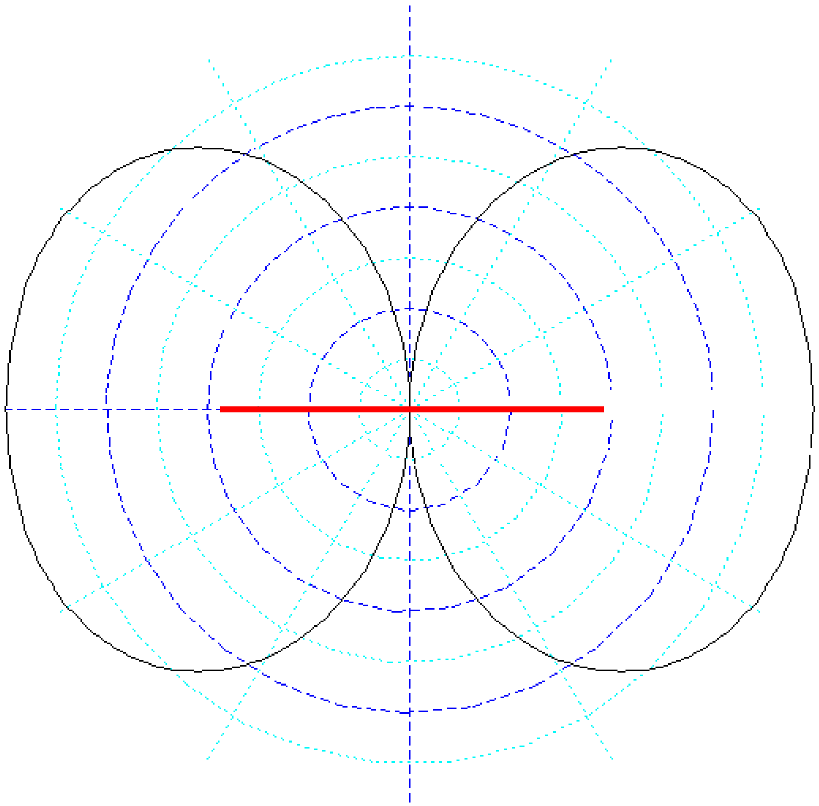


**8 dipoles**

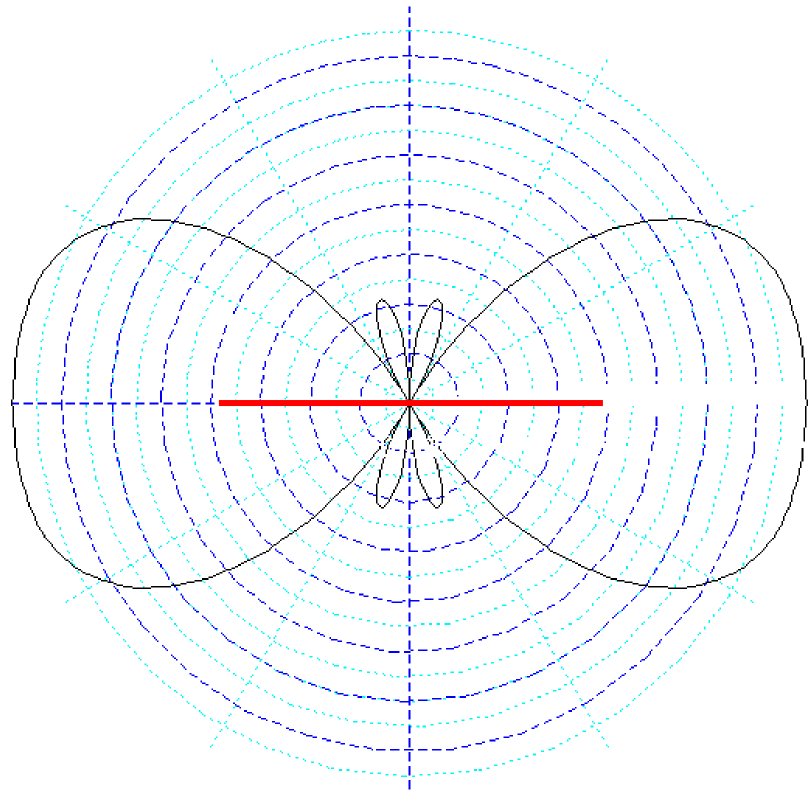


**16 dipoles**

# End-fire array (plots on the azimuthal plane, with $\theta = 90°$)

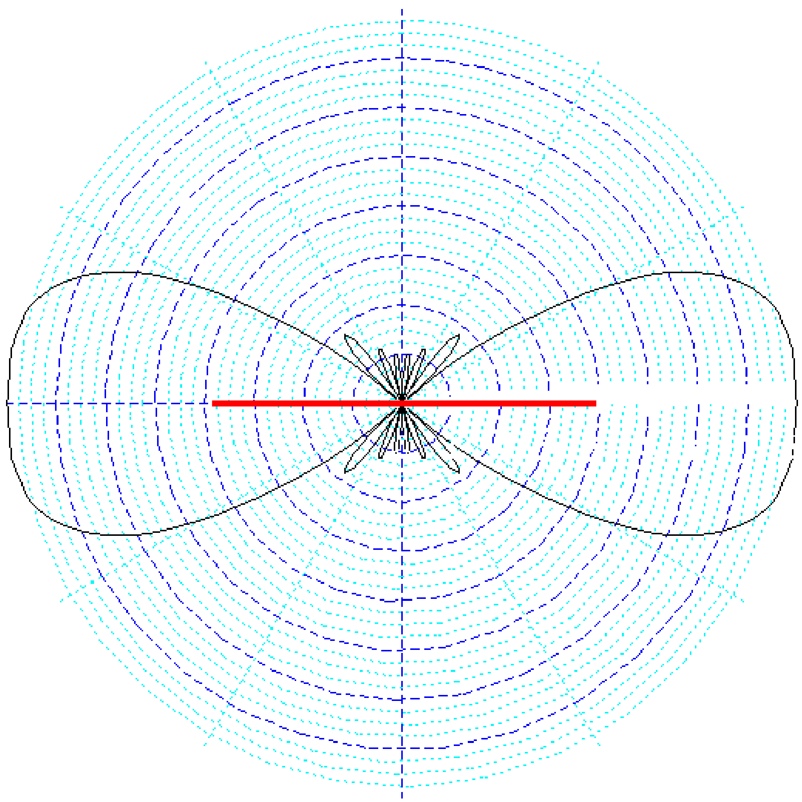


**2 dipoles**

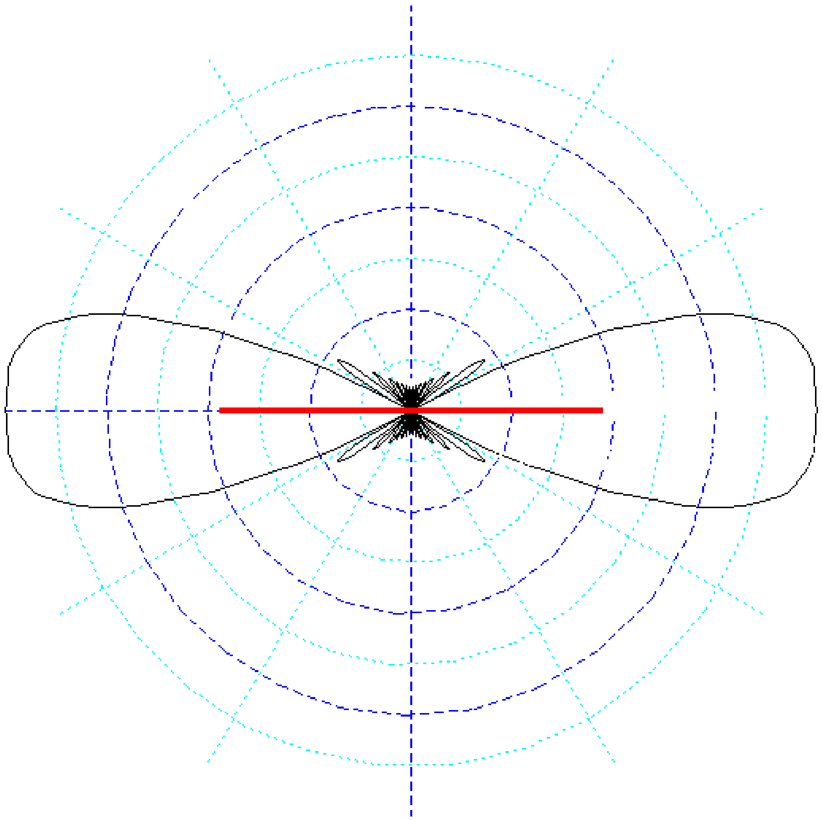


**4 dipoles**

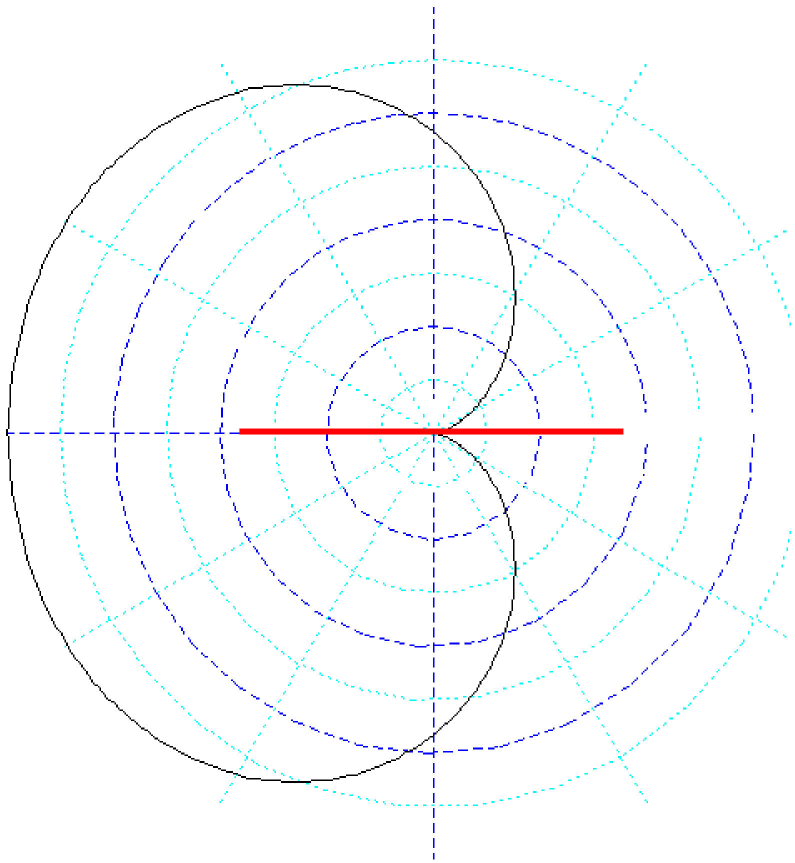


8 dipoles

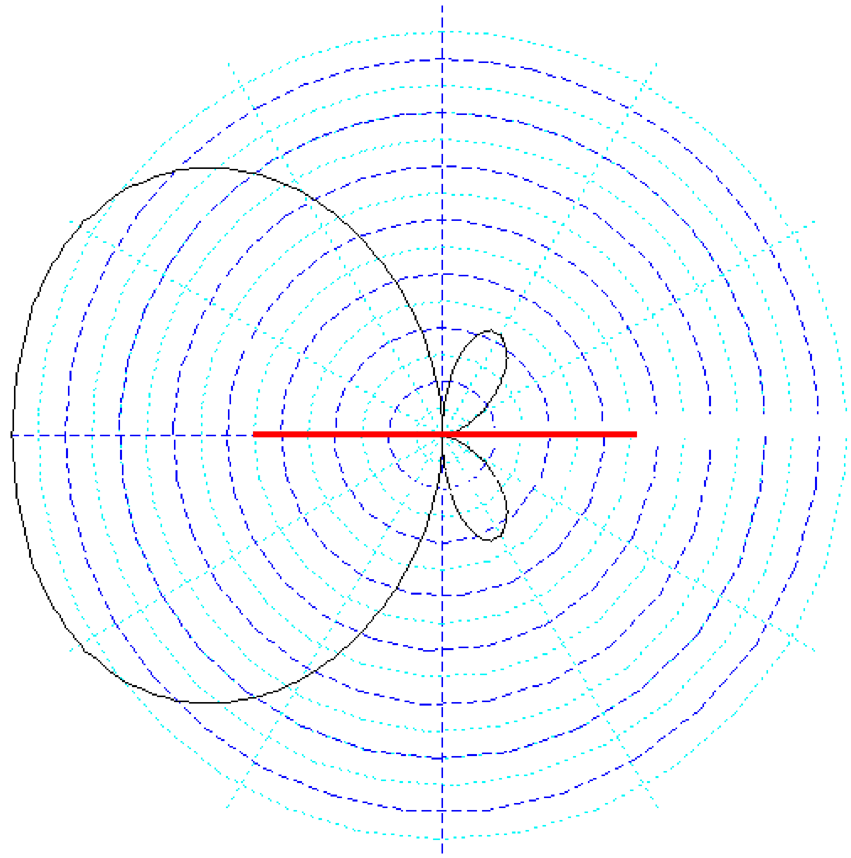


16 dipoles

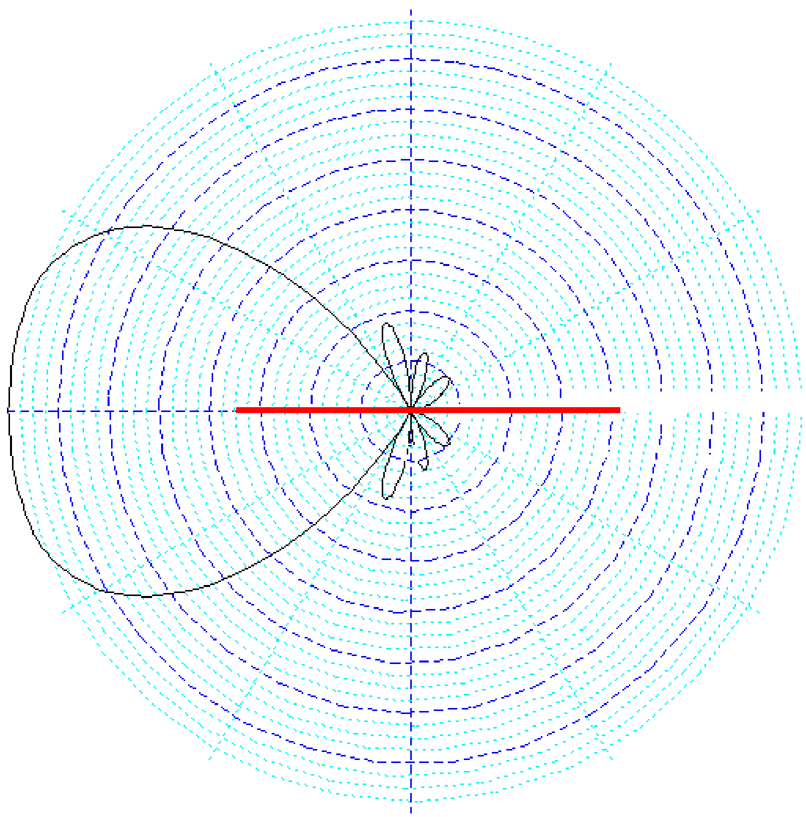
# Cardioid array (plots on the azimuthal plane, with $\theta = 90°$)



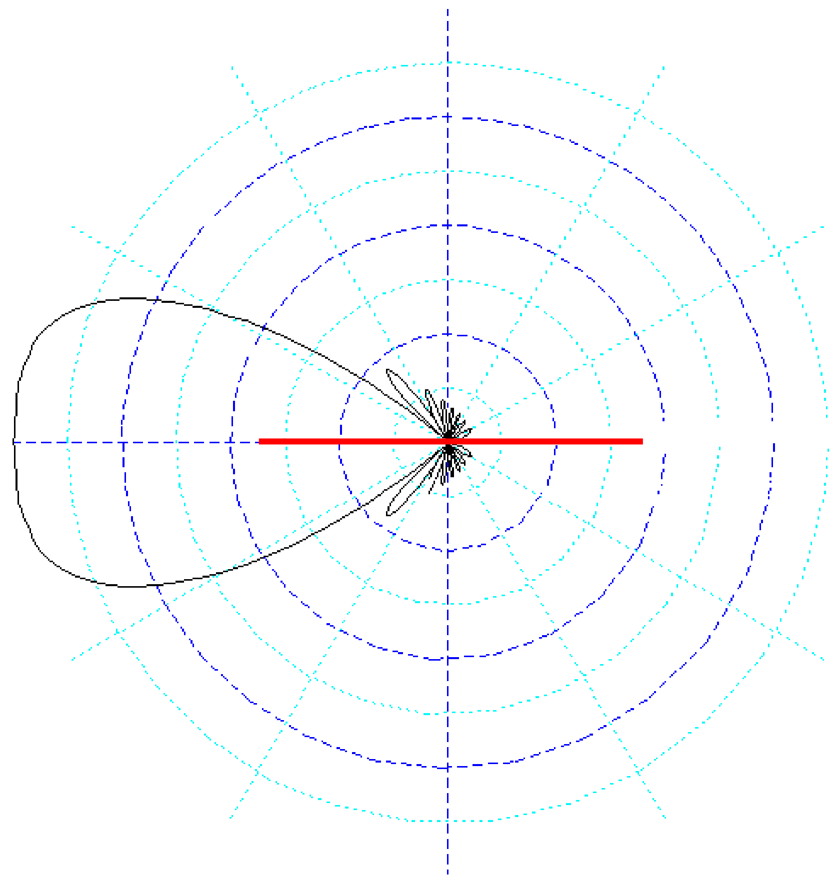
2 dipoles



4 dipoles



8 dipoles



16 dipoles