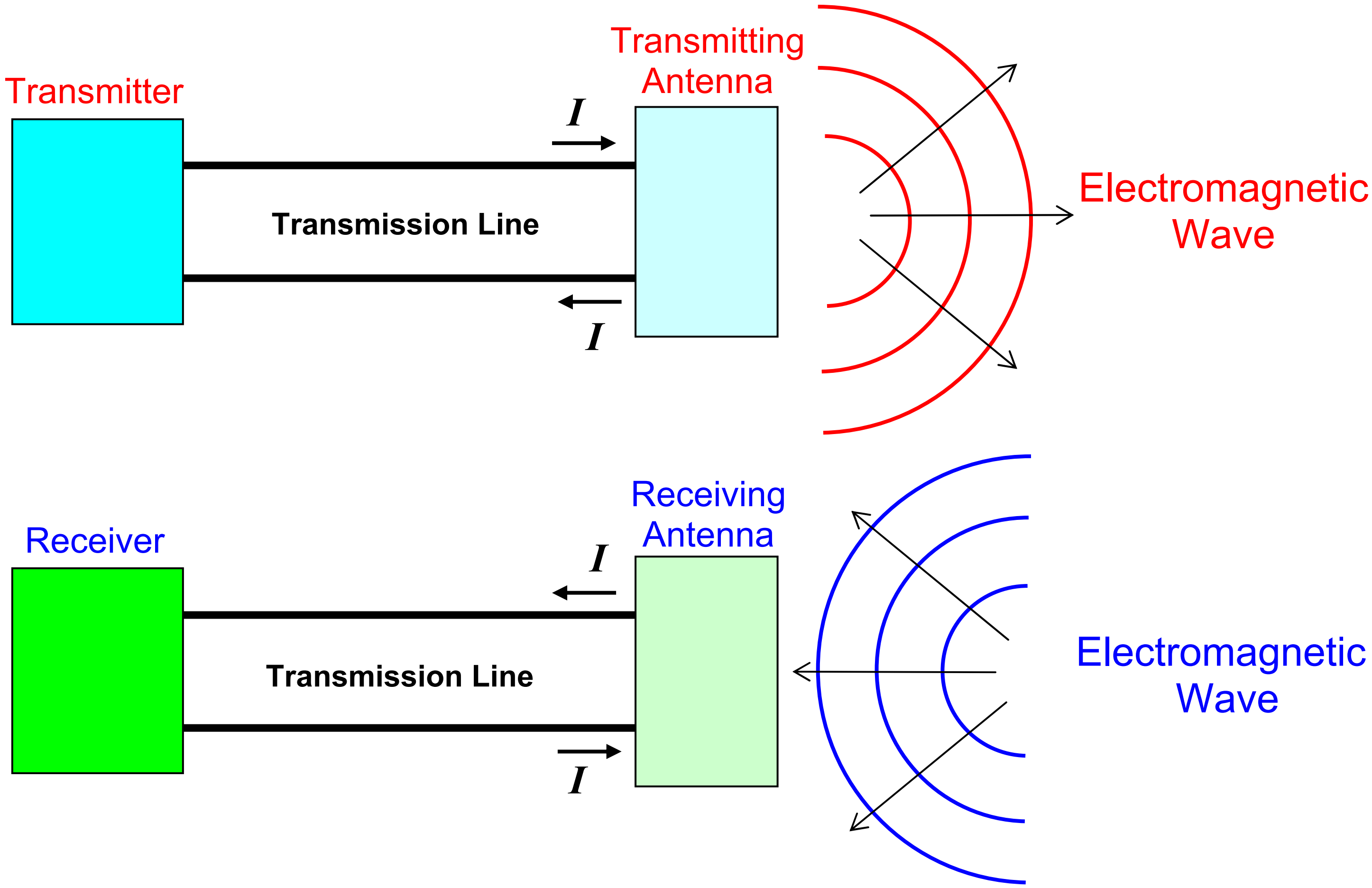
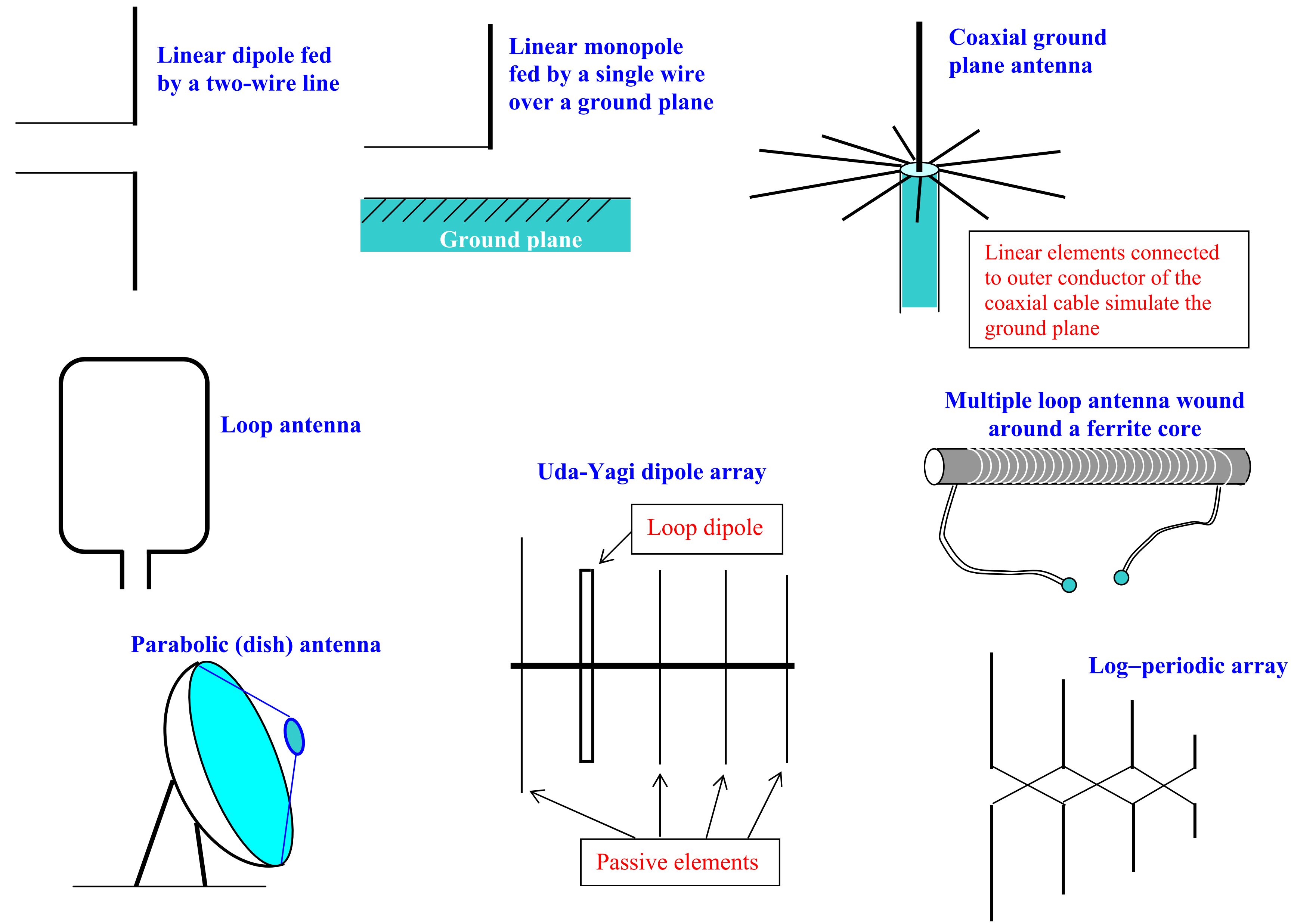
Antennas

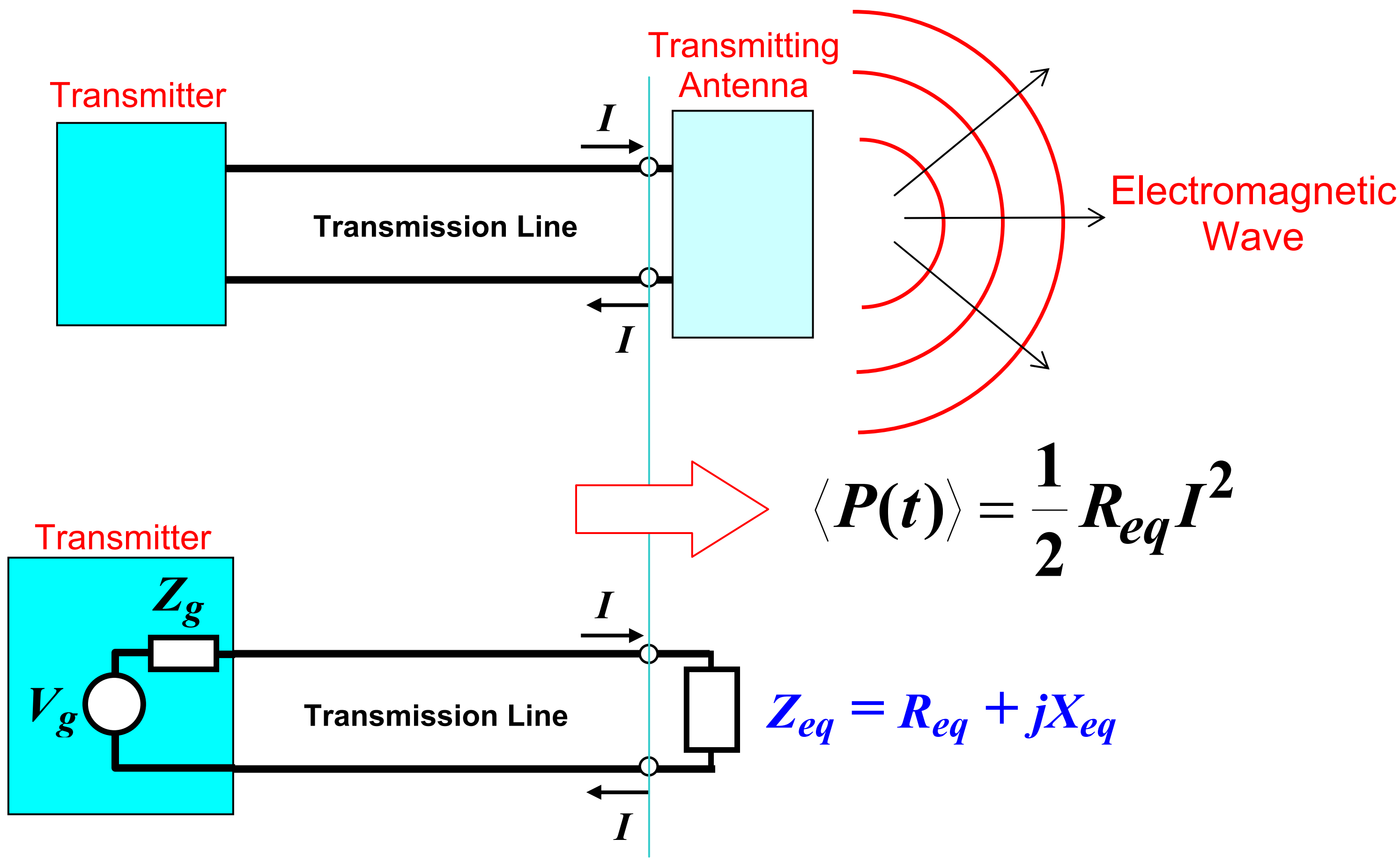


Antennas are transducers that transfer electromagnetic energy between a transmission line and free space.

Here are a few examples of common antennas:



From a circuit point of view, a transmitting antenna behaves like an equivalent impedance that dissipates the power transmitted

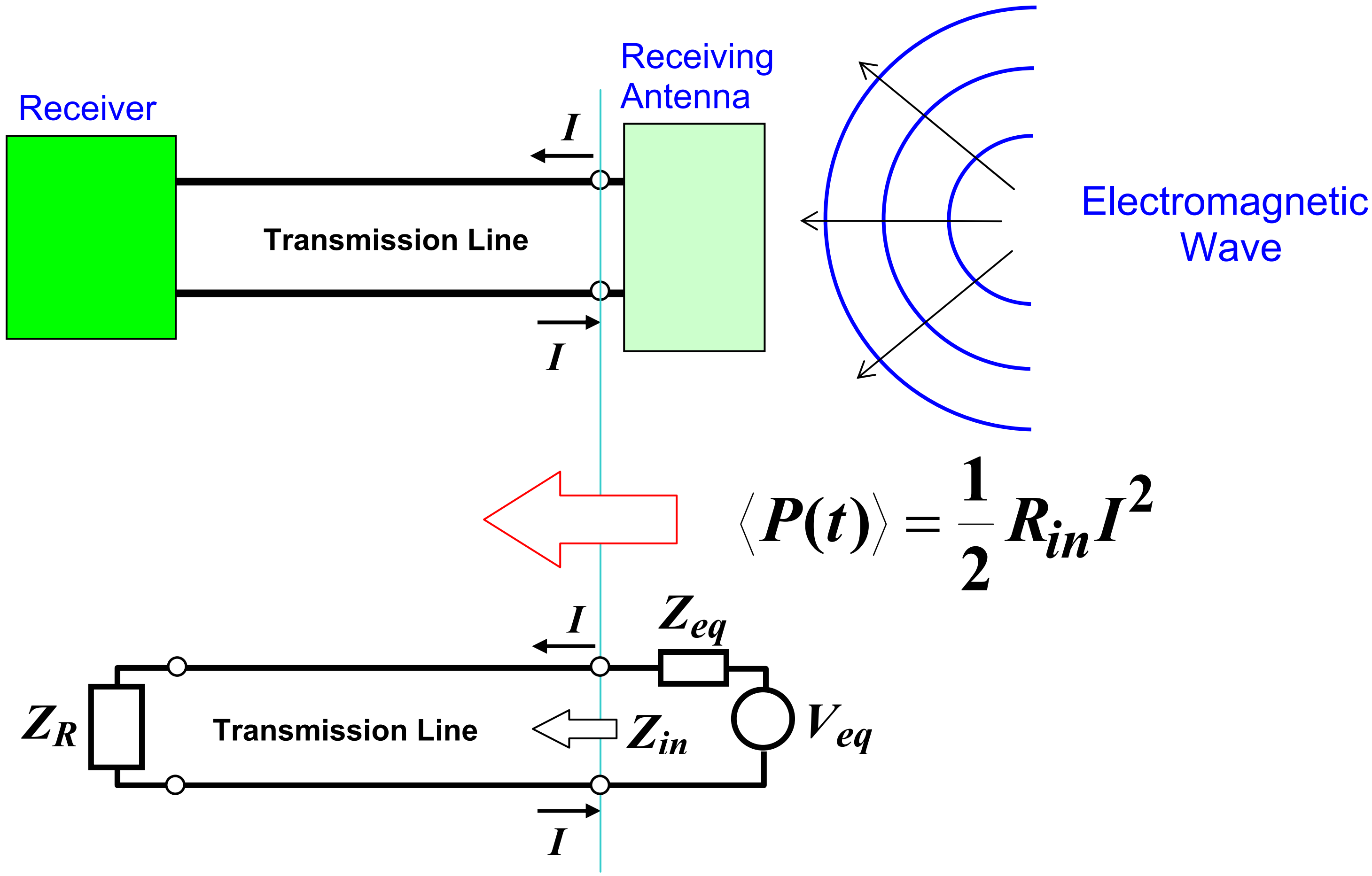


$$\langle P(t) \rangle = \frac{1}{2} R_{eq} I^2$$

$$Z_{eq} = R_{eq} + jX_{eq}$$

The transmitter is equivalent to a generator.

A receiving antenna behaves like a generator with an internal impedance corresponding to the antenna equivalent impedance.

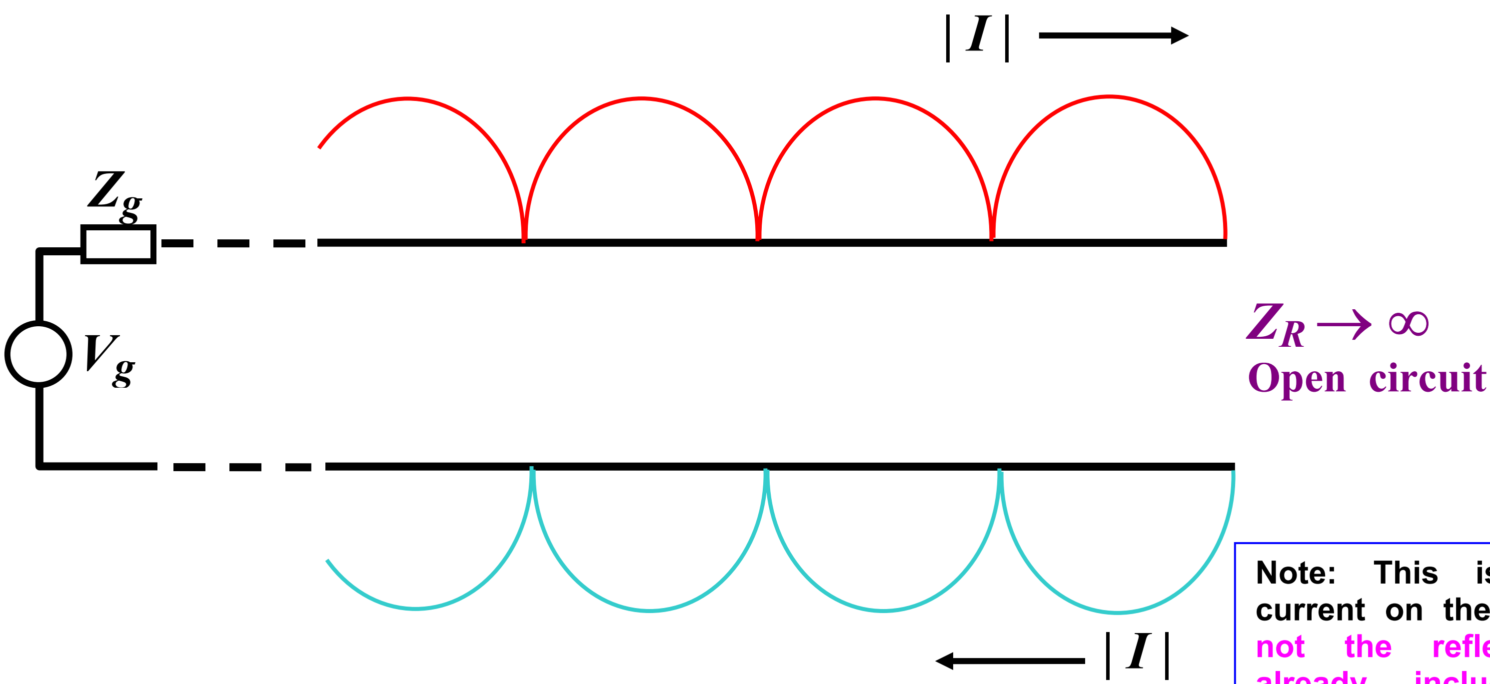


$$\langle P(t) \rangle = \frac{1}{2} R_{in} I^2$$

The receiver represents the load impedance that dissipates the time average power generated by the receiving antenna.

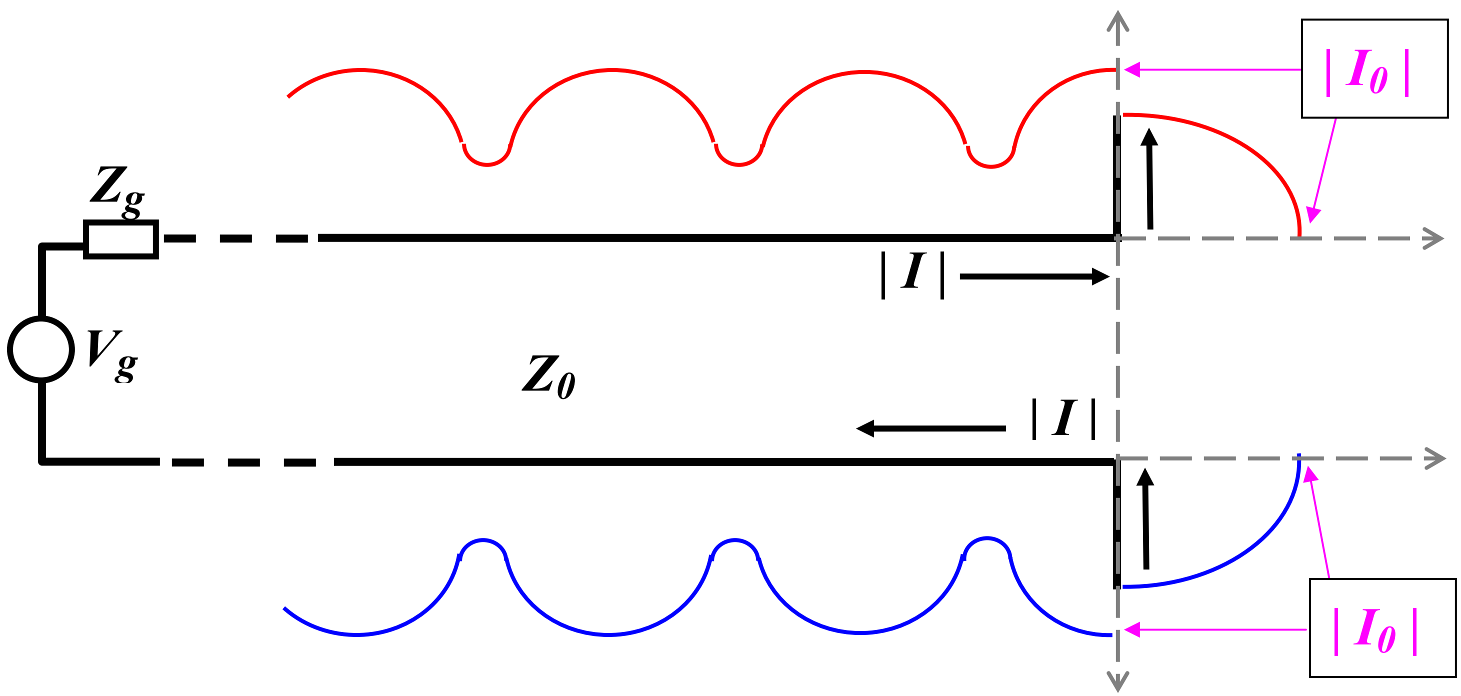
Antennas are in general reciprocal devices, which can be used both as transmitting and as receiving elements. This is how the antennas on cellular phones and walkie–talkies operate.

The basic principle of operation of an antenna is easily understood starting from a two–wire transmission line, terminated by an open circuit.

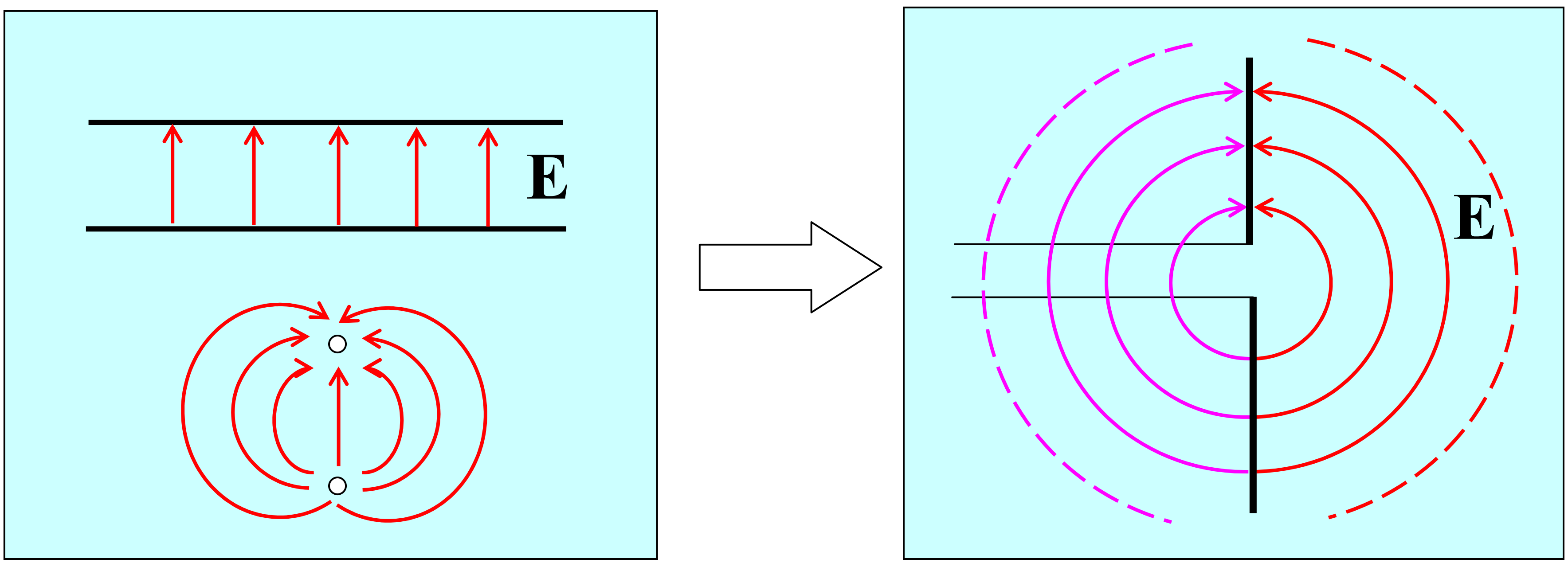


Note: This is the return current on the second wire, not the reflected current already included in the standing wave pattern.

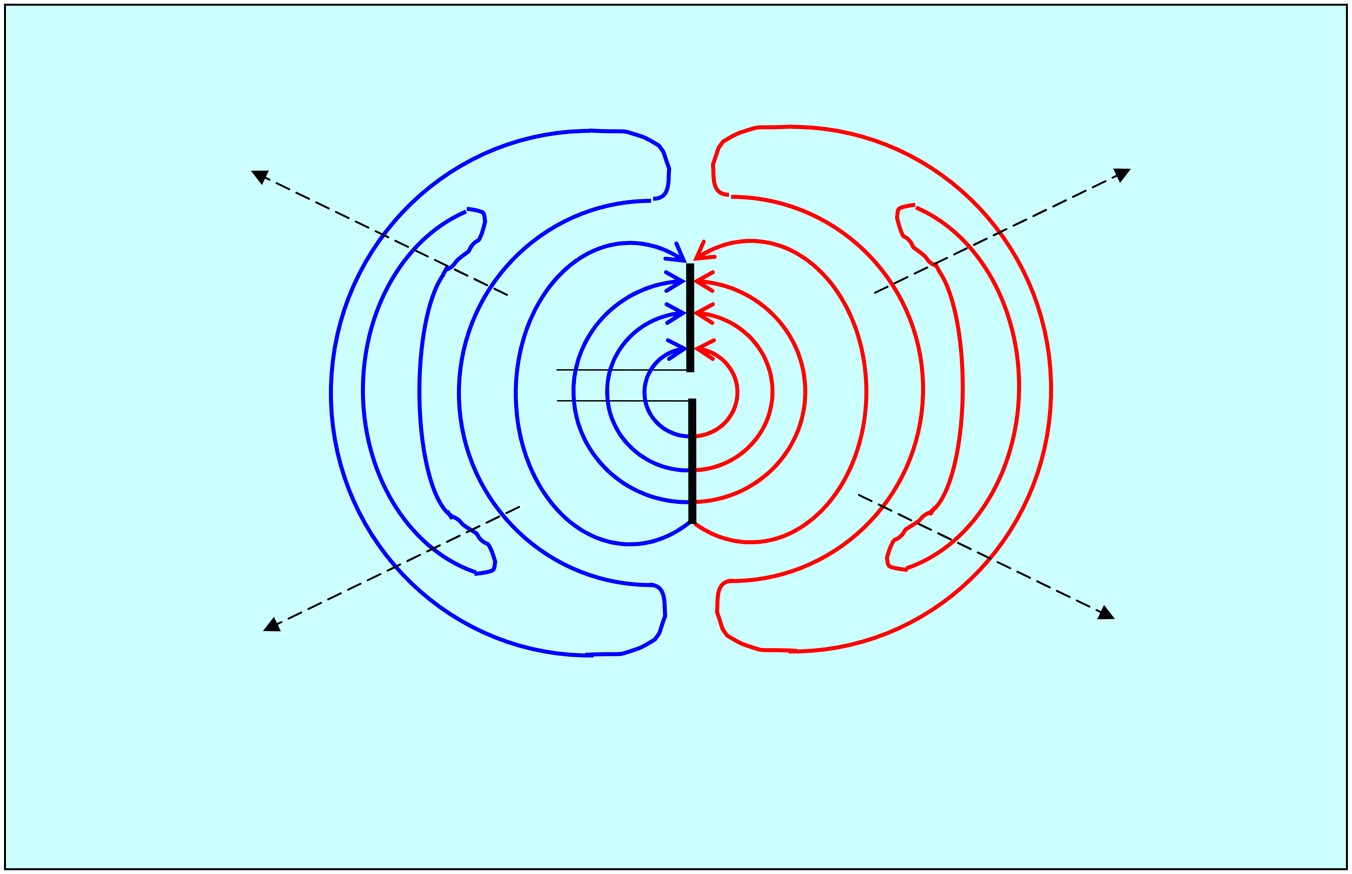
Imagine to bend the end of the transmission line, forming a dipole antenna. Because of the change in geometry, there is now an abrupt change in the characteristic impedance at the transition point, where the current is still continuous. The dipole leaks electromagnetic energy into the surrounding space, therefore it reflects less power than the original open circuit ⇒ the standing wave pattern on the transmission line is modified



In the space surrounding the dipole we have an electric field. At zero frequency (d.c. bias), fixed electrostatic field lines connect the metal elements of the antenna, with circular symmetry.



At higher frequency, the current oscillates in the wires and the field emanating from the dipole changes periodically. The field lines propagate away from the dipole and form closed loops.



The electromagnetic field emitted by an antenna obeys Maxwell's equations

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\vec{\mathbf{H}}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\varepsilon\vec{\mathbf{E}}$$

Under the assumption of uniform isotropic medium we have the wave equation:

$$\nabla \times \nabla \times \vec{\mathbf{E}} = -j\omega\mu\nabla \times \vec{\mathbf{H}} = -j\omega\mu\vec{\mathbf{J}} + \omega^2\mu\varepsilon\vec{\mathbf{E}}$$

$$\nabla \times \nabla \times \vec{\mathbf{H}} = \nabla \times \vec{\mathbf{J}} + j\omega\varepsilon\nabla \times \vec{\mathbf{E}}$$

$$= \nabla \times \vec{\mathbf{J}} + \omega^2\mu\varepsilon\vec{\mathbf{H}}$$

Note that in the regions with electrical charges ρ

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla\nabla \cdot \vec{\mathbf{E}} - \nabla^2\vec{\mathbf{E}} = \nabla(\rho/\varepsilon) - \nabla^2\vec{\mathbf{E}}$$

In general, these wave equations are difficult to solve, because of the presence of the terms with current and charge. It is easier to use the magnetic vector potential and the electric scalar potential.

The definition of the magnetic vector potential is

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$$

Note that since the divergence of the curl of a vector is equal to zero we always satisfy the zero divergence condition

$$\nabla \cdot \vec{\mathbf{B}} = \nabla \cdot \left(\nabla \times \vec{\mathbf{A}} \right) = \mathbf{0}$$

We have also

$$\nabla \times \vec{\mathbf{E}} = -j\omega\,\mu\,\vec{\mathbf{H}} = -j\omega\,\nabla \times \vec{\mathbf{A}} \quad \Rightarrow \quad \nabla \times \left(\vec{\mathbf{E}} + j\omega \vec{\mathbf{A}} \right) = \mathbf{0}$$

We define the scalar potential ϕ first noticing that

$$\nabla \times \left(\pm \nabla \phi \right) = \mathbf{0}$$

and then choosing (with sign convention as in electrostatics)

$$\nabla \times \left(\vec{\mathbf{E}} + j\omega \vec{\mathbf{A}} \right) = \nabla \times \left(-\nabla \phi \right) \quad \Rightarrow \quad \vec{\mathbf{E}} = -j\omega \vec{\mathbf{A}} - \nabla \phi$$

Note that the magnetic vector potential is not uniquely defined, since for any arbitrary scalar field ψ

$$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}} = \nabla \times \left(\vec{\mathbf{A}} + \nabla \psi \right)$$

In order to uniquely define the magnetic vector potential, the standard approach is to use the Lorenz gauge

$$\nabla \cdot \vec{\mathbf{A}} + j\omega \mu \varepsilon \phi = \mathbf{0}$$

From Maxwell's equations

$$\nabla \times \vec{\mathbf{H}} = \frac{1}{\mu} \nabla \times \vec{\mathbf{B}} = \vec{\mathbf{J}} + j\omega\,\varepsilon\,\vec{\mathbf{E}}$$

$$\nabla \times \vec{\mathbf{B}} = \mu\,\vec{\mathbf{J}} + j\omega\,\mu\,\varepsilon\,\vec{\mathbf{E}}$$

$$\Rightarrow \quad \nabla \times \left(\nabla \times \vec{\mathbf{A}}\right) = \mu\,\vec{\mathbf{J}} + j\omega\,\mu\,\varepsilon\left(-j\omega\vec{\mathbf{A}} - \nabla\phi\right)$$

From vector calculus

$$\nabla \times \left(\nabla \times \ldots\right) = \nabla\left(\nabla \cdot \ \ldots\right) - \nabla^2 \ldots$$

$$\nabla \times \left(\nabla \times \vec{\mathbf{A}}\right) = \nabla\left(\nabla \cdot \vec{\mathbf{A}}\right) - \nabla^2\vec{\mathbf{A}} = \mu\,\vec{\mathbf{J}} + \omega^2\,\mu\,\varepsilon\,\vec{\mathbf{A}} - j\omega\,\mu\,\varepsilon\,\nabla\phi$$

Lorenz Gauge

$$\nabla \cdot \vec{\mathbf{A}} = -j\omega\,\mu\,\varepsilon\,\phi \quad \Rightarrow \quad \nabla\left(\nabla \cdot \vec{\mathbf{A}}\right) = -j\omega\,\mu\,\varepsilon\nabla\phi$$

Finally, the wave equation for the magnetic vector potential is

$$\nabla^2 \vec{\mathbf{A}} + \omega^2 \mu \varepsilon \vec{\mathbf{A}} = \nabla^2 \vec{\mathbf{A}} + \beta^2 \vec{\mathbf{A}} = -\mu \vec{\mathbf{J}}$$

For the electric field we have

$$\nabla \cdot \vec{\mathbf{D}} = \rho \quad \Rightarrow \quad \nabla \cdot \vec{\mathbf{E}} = \nabla \cdot \left(-j\omega \vec{\mathbf{A}} - \nabla \phi \right) = \frac{\rho}{\varepsilon}$$

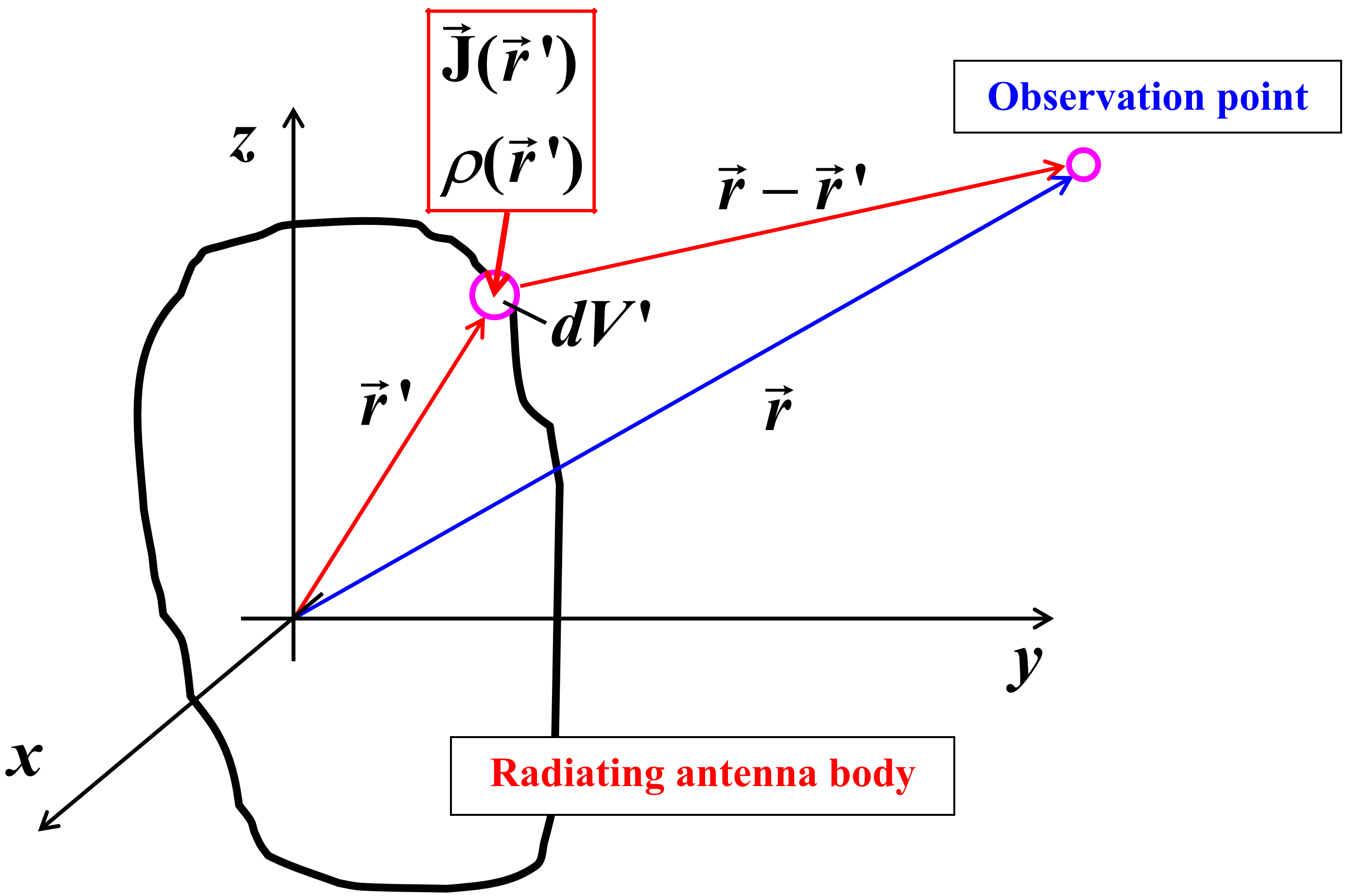
$$\nabla^2 \phi + j\omega \nabla \cdot \vec{\mathbf{A}} = \nabla^2 \phi + j\omega \left(-j\omega \mu \varepsilon \phi \right) = -\frac{\rho}{\varepsilon}$$

The wave equation for the electric scalar potential is

$$\nabla^2 \phi + \omega^2 \mu \varepsilon \phi = \nabla^2 \phi + \beta^2 \phi = -\frac{\rho}{\varepsilon}$$

The wave equations are inhomogenoeous Helmholtz equations, which apply to regions where currents and charges are not zero.

We use the following system of coordinates for an antenna body



The generals solutions for the wave equations are

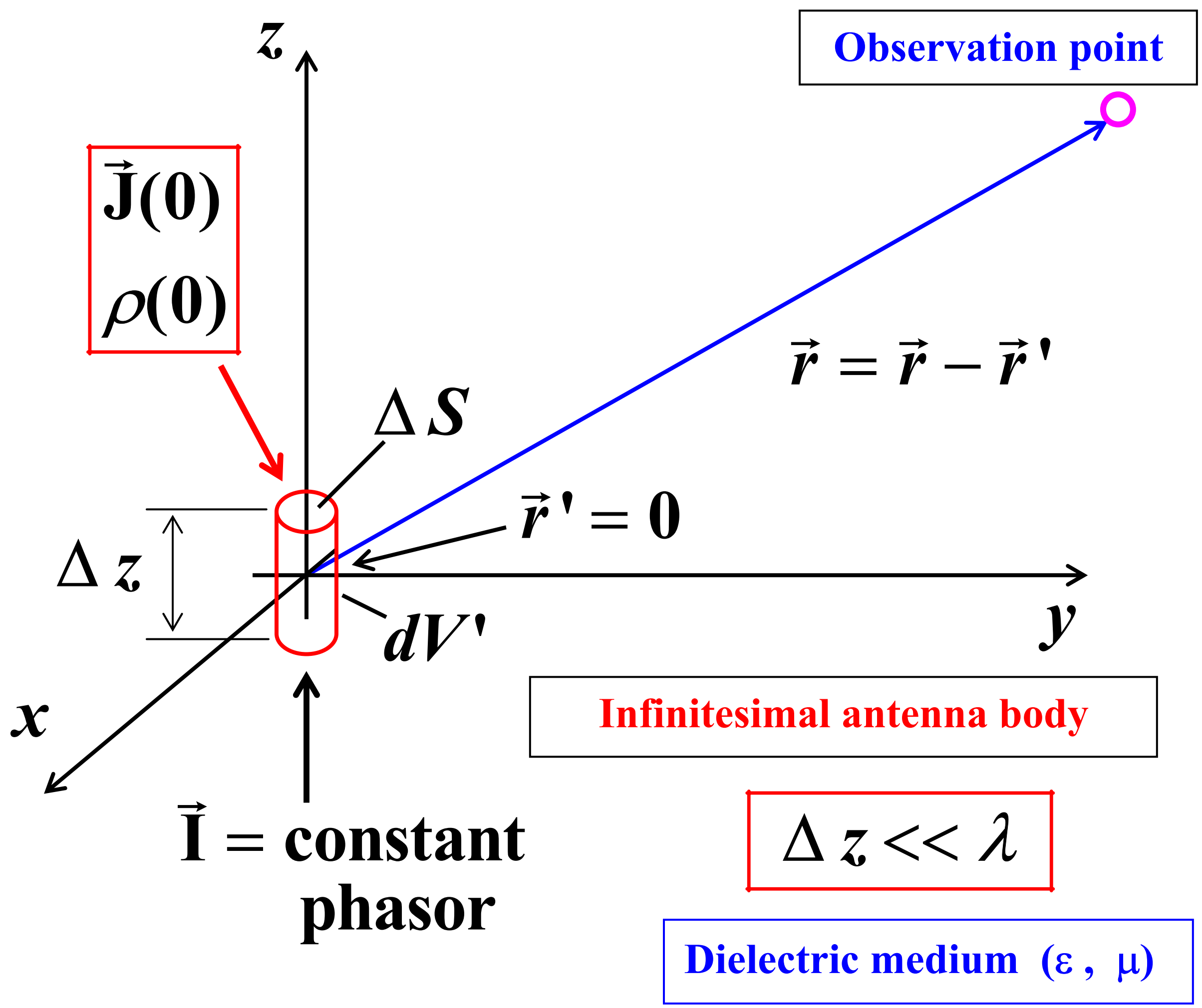
$$\vec{\mathbf{A}}(\vec{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\vec{J}(\vec{r}\,')\, e^{-j\beta|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \quad dV'$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon} \iiint_V \frac{\rho(\vec{r}\,')\, e^{-j\beta|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \quad dV'$$

The integrals are extended to all points over the antenna body where the sources (current density, charge) are not zero. The effect of each volume element of the antenna is to radiate a radial wave

$$\frac{e^{-j\beta|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

Infinitesimal Antenna



The current flowing in the infinitesimal antenna is assumed to be constant and oriented along the *z*–axis

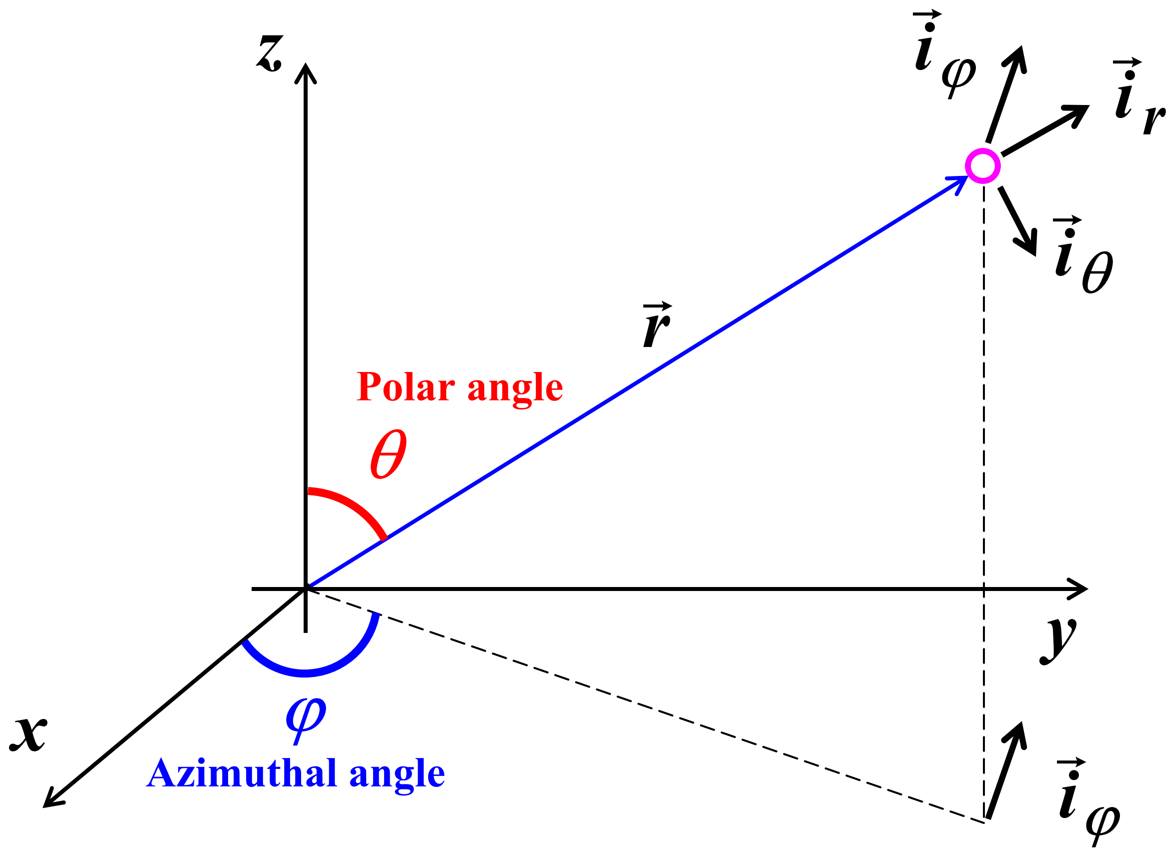
$$\vec{\mathbf{I}} = \Delta S \cdot \vec{\mathbf{J}}(\vec{r}\,') = \Delta S \cdot \vec{\mathbf{J}}(0) \qquad \Delta V' = \Delta S \cdot \Delta z$$

$$\Delta V' \vec{\mathbf{J}}(\vec{r}\,') = \left|\vec{\mathbf{I}}\right| \Delta z \; \vec{i}_z$$

The solution of the wave equation for the magnetic vector potential simply becomes the evaluation of the integrand at the origin

$$\vec{\mathbf{A}} = \frac{\mu \left|\vec{\mathbf{I}}\right| \Delta z \, e^{-j\beta r}}{4\pi r} \; \vec{i}_z \quad \Rightarrow \quad \begin{cases} \vec{\mathbf{H}} = \dfrac{1}{\mu} \nabla \times \vec{\mathbf{A}} \\[2ex] \vec{\mathbf{E}} = \dfrac{1}{j\omega\varepsilon} \nabla \times \vec{\mathbf{H}} \end{cases}$$

There is still a major mathematical step left. The curl operations must be expressed in terms of spherical coordinates



In spherical coordinates

$$\nabla \times \vec{\mathbf{A}} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{i}_r & r\,\vec{i}_\theta & r\sin\theta\,\vec{i}_\varphi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{\partial}{\partial\varphi}\left(A_\theta\right)\right]\vec{i}_r$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(A_r\right) - \frac{\partial}{\partial r}\left(r\,A_\varphi\right)\right]\vec{i}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,A_\theta\right) - \frac{\partial}{\partial\theta}\left(A_r\right)\right]\vec{i}_\varphi$$

We had

$$\vec{\mathbf{A}} = \frac{\mu \left|\vec{\mathbf{I}}\right| \Delta z \, e^{-j\beta r}}{4\pi r} \vec{i}_z \qquad \textit{with} \qquad \vec{i}_z = \vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta$$

$$\Rightarrow \quad \boxed{\nabla \times \vec{\mathbf{A}} = \vec{i}_\varphi \frac{j \mu \beta \left|\vec{\mathbf{I}}\right| \Delta z \, e^{-j\beta r}}{4\pi r} \left(1 + \frac{1}{j\beta r}\right) \sin\theta}$$

For the fields we have

$$\boxed{\vec{\mathbf{H}} = \frac{1}{\mu} \nabla \times \vec{\mathbf{A}} = \vec{i}_\varphi \frac{j\beta \left|\vec{\mathbf{I}}\right| \Delta z \, e^{-j\beta r}}{4\pi r} \left(1 + \frac{1}{j\beta r}\right) \sin\theta}$$

$$\vec{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla\times\vec{\mathbf{H}} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{j\beta\left|\vec{\mathbf{I}}\right|\Delta z\, e^{-j\beta r}}{4\pi r}$$

$$\times\left[2\cos\theta\left(\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\vec{i}_r\right.$$

$$\left.+\sin\theta\left(1+\frac{1}{j\beta r}+\frac{1}{(j\beta r)^2}\right)\vec{i}_\theta\right]$$

The general field expressions can be simplified for observation point at large distance from the infinitesimal antenna

$$1 >> \left|\frac{1}{j\beta r}\right| >> \left|\frac{1}{(j\beta r)^2}\right| \quad \Rightarrow \quad \beta r = \frac{2\pi}{\lambda} r >> 1$$

At large distance we have the expressions for the Far Field

$$\vec{\mathbf{H}} \approx \vec{i}_{\varphi} \, \frac{j\beta \left| \vec{\mathbf{I}} \right| \Delta z \, e^{-j\beta r}}{4\pi r} \sin\theta$$

$$\vec{\mathbf{E}} \approx \vec{i}_{\theta} \sqrt{\frac{\mu}{\varepsilon}} \, \frac{j\beta \left| \vec{\mathbf{I}} \right| \Delta z \, e^{-j\beta r}}{4\pi r} \sin\theta$$

$$2\pi r >> \lambda$$

- **At sufficient distance from the antenna, the radiated fields are perpendicular to each other and to the direction of propagation.**
- **The magnetic field and electric field are in phase and**

$$\left| \vec{\mathbf{E}} \right| = \sqrt{\frac{\mu}{\varepsilon}} \left| \vec{\mathbf{H}} \right| = \eta \left| \vec{\mathbf{H}} \right|$$

These are also properties of uniform plane waves.

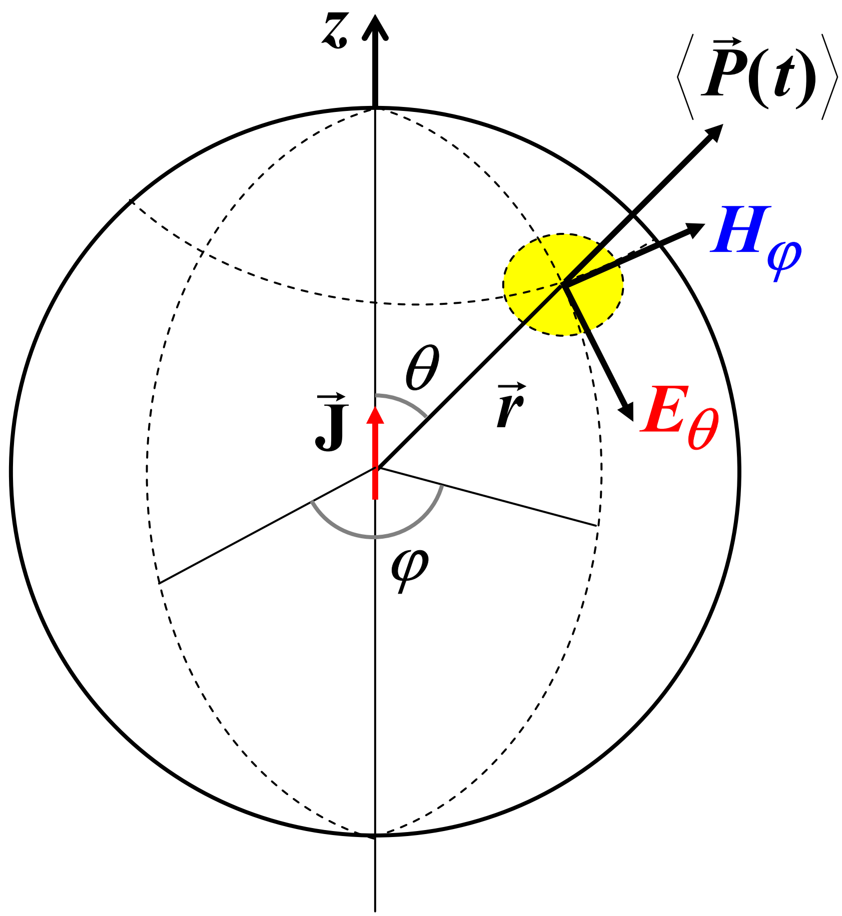
However, there are significant differences with respect to a uniform plane wave:

- **The surfaces of constant phase are spherical instead of planar, and the wave travels in the radial direction.**
- **The intensities of the fields are inversely proportional to the distance, therefore the field intensities decay while they are constant for a uniform plane wave.**
- **The field intensities are not constant on a given surface of constant phase. The intensity depends on the sine of the polar angle θ.**

The radiated power density is

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left\{ \vec{\mathbf{E}} \times \vec{\mathbf{H}}^{*} \right\} = \vec{i}_r \, \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}} \left| H_\varphi \right|^2$$

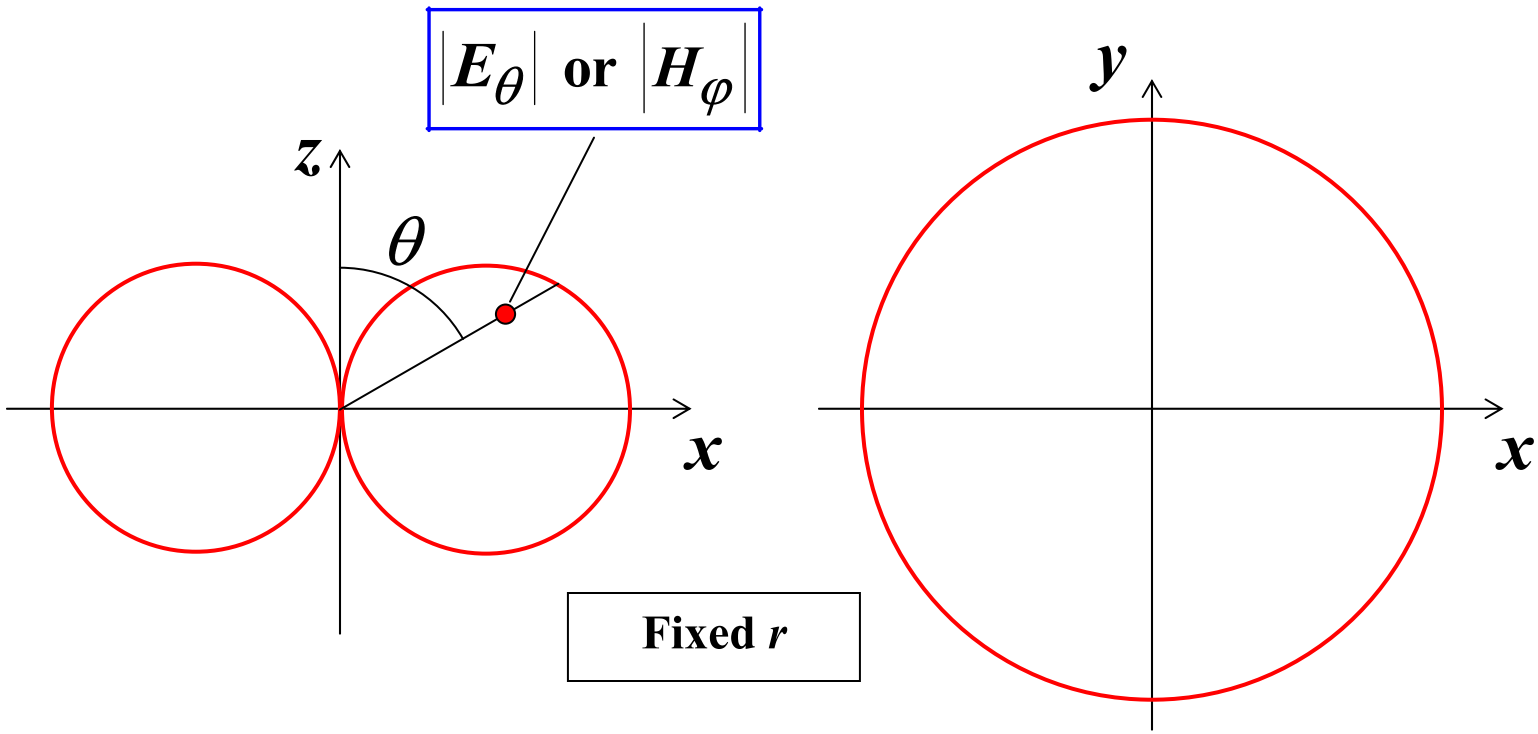
$$= \vec{i}_r \, \frac{\eta}{2}\left(\frac{\beta \left| \vec{\mathbf{I}} \right| \Delta z}{4\pi r} \right)^2 \sin^2\theta$$



The spherical wave resembles a plane wave *locally* in a small neighborhood of the point (r, θ, φ).

Radiation Patterns

Electric Field and Magnetic Field



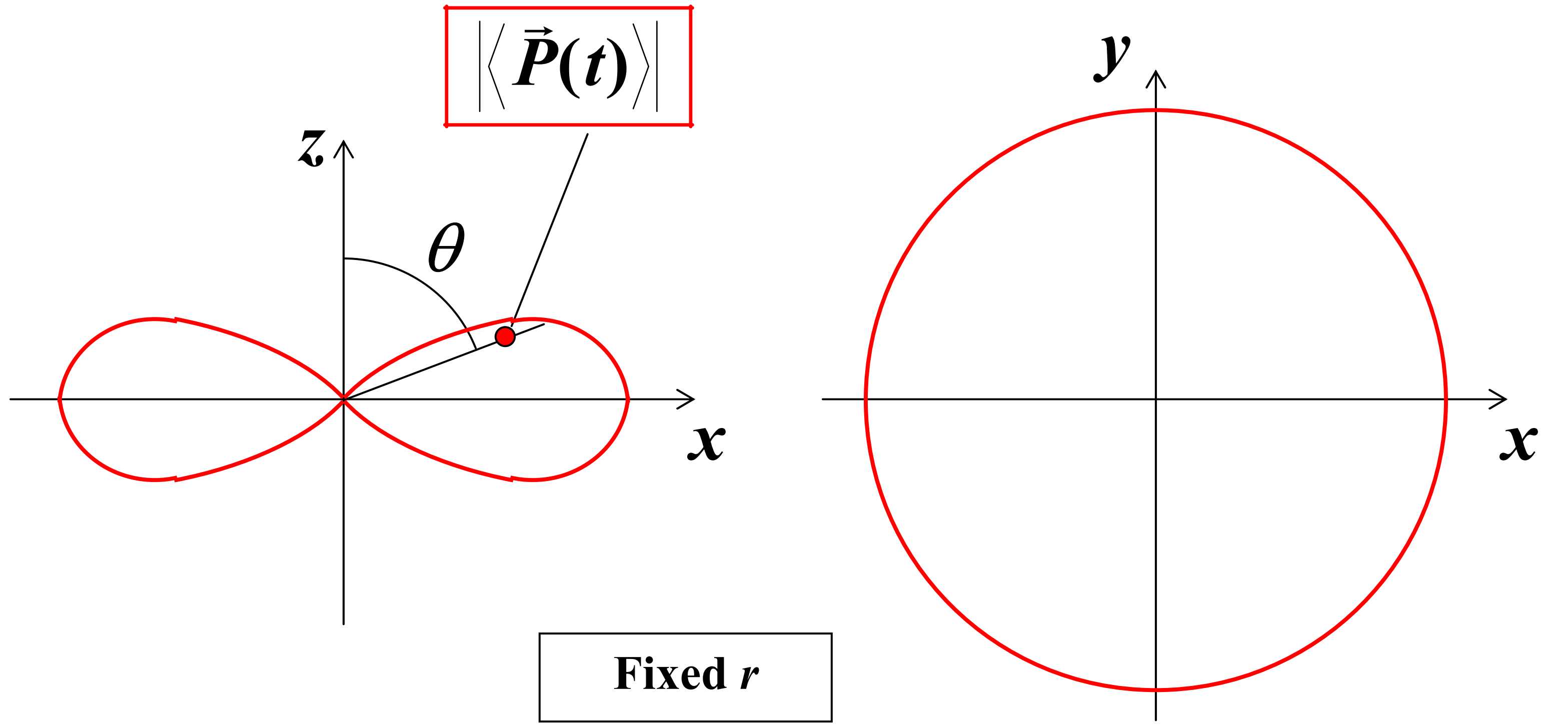
Plane containing the antenna

proportional to $\sin\theta$

Plane perpendicular to the antenna

omnidirectional or isotropic

Time–average Power Flow (Poynting Vector)



Fixed *r*

Plane containing the antenna

proportional to $\sin^2\theta$

Plane perpendicular to the antenna

omnidirectional or isotropic

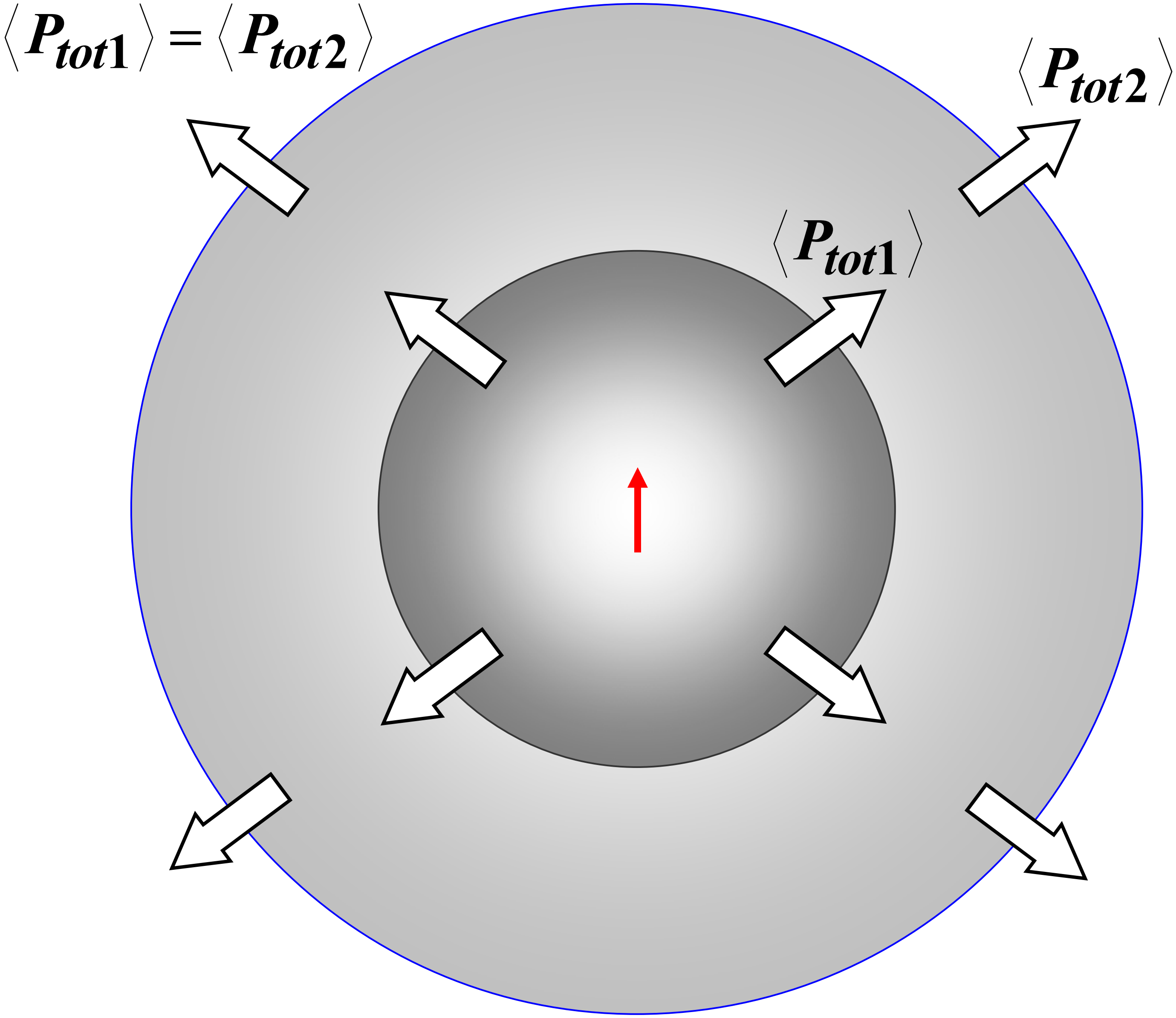
Total Radiated Power

The time–average power flow is not uniform on the spherical wave front. In order to obtain the total power radiated by the infinitesimal antenna, it is necessary to integrate over the sphere

$$\langle P_{tot} \rangle = \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} \quad \int_0^{\pi} d\theta \, r^2 \sin\theta \left| \langle \vec{P}(t) \rangle \right|$$

$$= \frac{\eta}{2} \left(\frac{\beta \left| \vec{\mathrm{I}} \right| \Delta z}{4\pi r} \right)^2 2\pi \, r^2 \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{=4/3} \quad = \frac{4\pi\eta}{3} \left(\frac{\beta \left| \vec{\mathrm{I}} \right| \Delta z}{4\pi} \right)^2$$

Note: the total radiated power is independent of distance. Although the power density decreases with distance, the integral of the power over concentric spherical wave fronts remains constant.


$\langle P_{tot1} \rangle = \langle P_{tot2} \rangle$
$\langle P_{tot2} \rangle$
$\langle P_{tot1} \rangle$

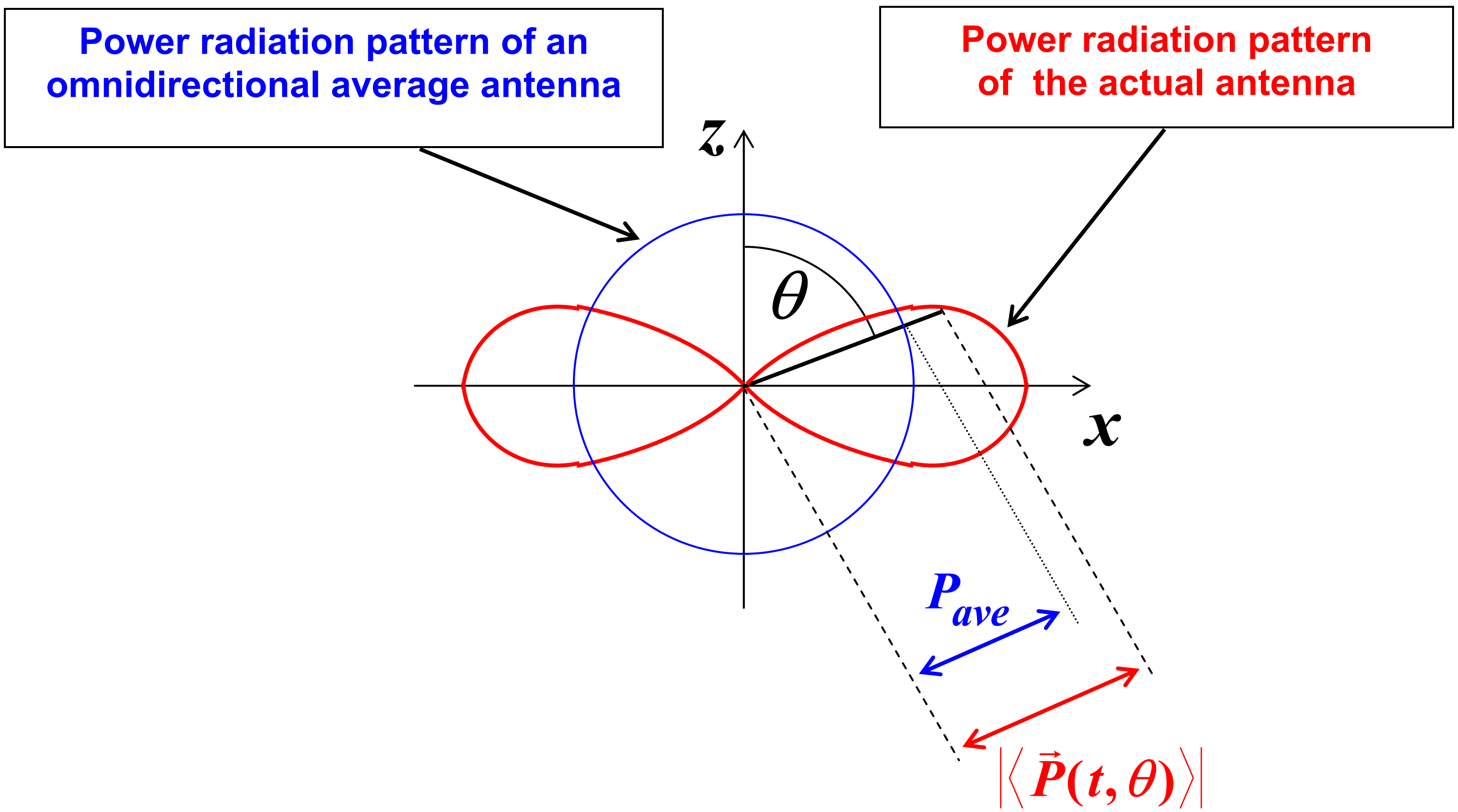
The total radiated power is also the power delivered by the transmission line to the real part of the equivalent impedance seen at the input of the antenna

$$\langle P_{tot} \rangle = \frac{1}{2}\left|\vec{\mathbf{I}}\right|^2 R_{eq} = \frac{4\pi\eta}{3}\left(\frac{2\pi}{\lambda}\frac{\left|\vec{\mathbf{I}}\right|\Delta z}{4\pi}\right)^2 = \frac{1}{2}\left|\vec{\mathbf{I}}\right|^2 \underbrace{\left[\frac{2\pi\eta}{3}\left(\frac{\Delta z}{\lambda}\right)^2\right]}_{R_{eq}}$$

The equivalent resistance of the antenna is usually called radiation resistance. In free space

$$\eta = \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi\,[\Omega] \quad \Rightarrow \quad R_{eq} = 80\pi^2\left(\frac{\Delta z}{\lambda}\right)^2 [\Omega]$$

The **total radiated power** is also used to define the **average** power density emitted by the antenna. The average power density corresponds to the radiation of a hypothetical **omnidirectional (isotropic) antenna**, which is used as a reference to understand the **directive** properties of any antenna.



The time–average power density is given by

$$P_{ave} = \frac{\textbf{Total Radiated Power}}{\textbf{Surface of wave front}} =$$

$$= \frac{\langle P_{tot} \rangle}{4\pi r^2} = \frac{\eta}{12\pi}\left(\beta \left|\vec{\mathbf{I}}\right| \Delta z\right)^2 \frac{1}{4\pi r^2} = \frac{\eta}{3}\left(\frac{\beta \left|\vec{\mathbf{I}}\right| \Delta z}{4\pi r}\right)^2$$

The directive gain of the infinitesimal antenna is defined as

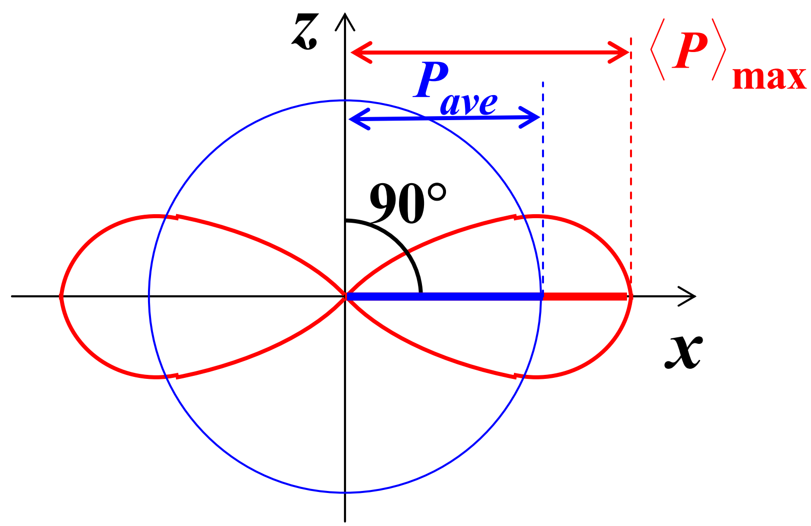
$$D(\theta,\varphi) = \frac{\left|\langle \vec{P}(t,r,\theta) \rangle\right|}{P_{ave}} = \frac{\eta}{2}\left(\frac{\beta \left|\vec{\mathbf{I}}\right| \Delta z}{4\pi r}\right)^2 \sin^2\theta \left(\frac{\eta}{3}\left(\frac{\beta \left|\vec{\mathbf{I}}\right| \Delta z}{4\pi r}\right)^2\right)^{-1}$$

$$= \frac{3}{2}\sin^2\theta$$

The maximum value of the directive gain is called directivity of the antenna. For the infinitesimal antenna, the maximum of the directive gain occurs when the polar angle θ is 90°

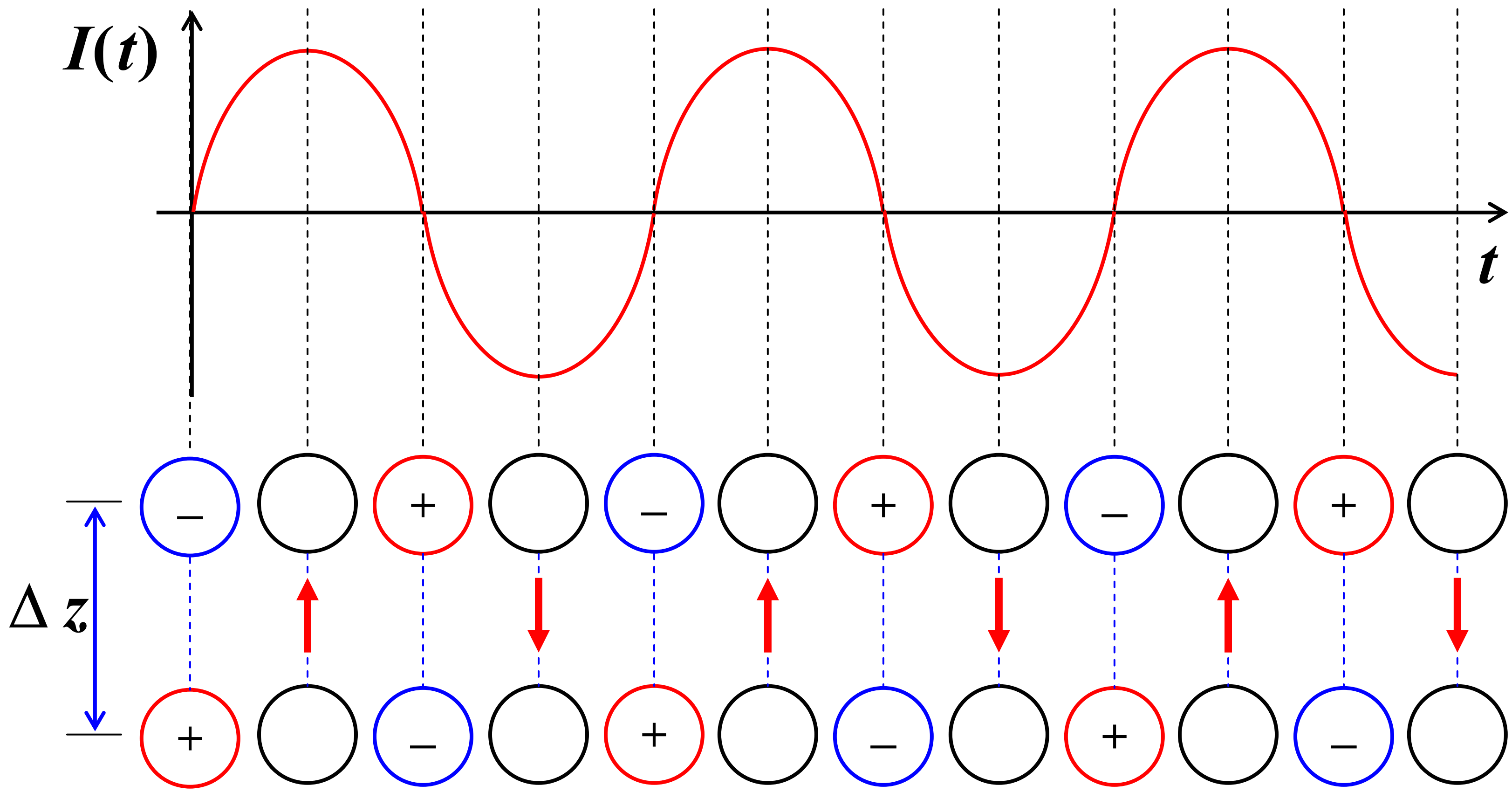
$$\text{Directivity} = \max\left\{ D(\theta,\varphi)\right\} = \frac{3}{2}\sin^2\left(\frac{\pi}{2}\right) = 1.5$$

The directivity gives a measure of how the actual antenna performs in the direction of maximum radiation, with respect to the ideal isotropic antenna which emits the average power in all directions.



The infinitesimal antenna is a suitable model to study the behavior of the elementary radiating element called Hertzian dipole.

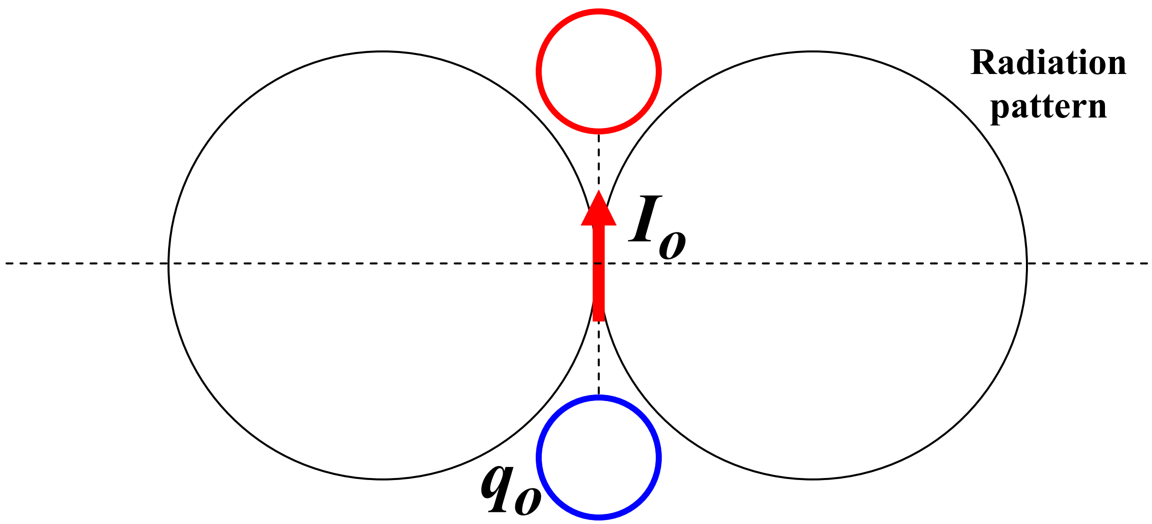
Consider two small charge reservoirs, separated by a distance Δz, which exchange mobile charge in the form of an oscillatory curent



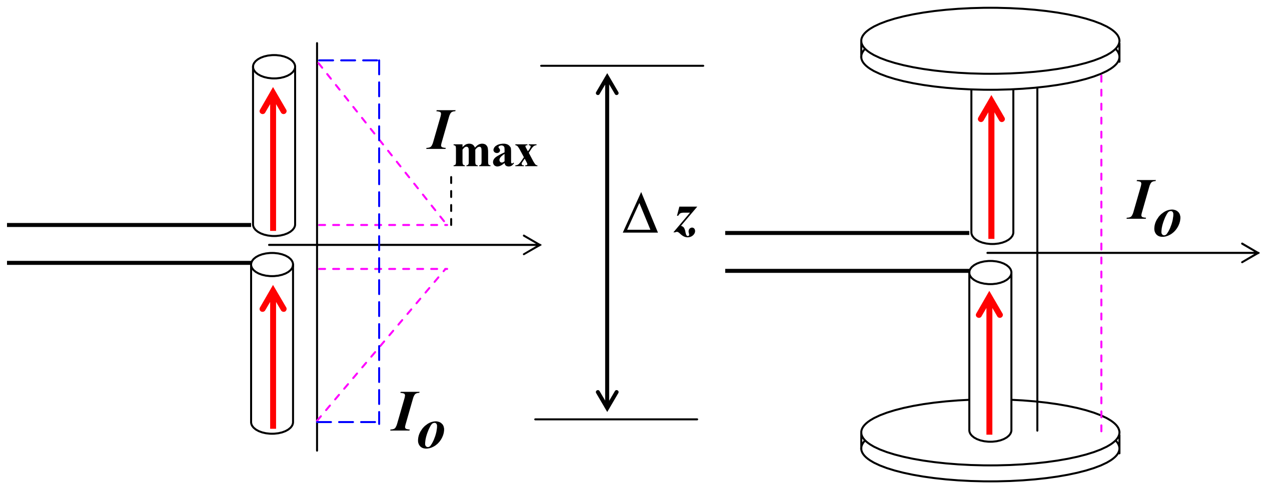
The Hertzian dipole can be used as an elementary model for many natural charge oscillation phenomena. The radiated fields can be described by using the results of the infinitesimal antenna.

Assuming a sinusoidally varying charge flow between the reservoirs, the oscillating current is

$$\underbrace{I(t)}_{\text{current flowing out of reservoir}} = \frac{d}{dt}q(t) = \frac{d}{dt}\underbrace{q_o \cos(\omega t)}_{\text{charge on reference reservoir}} \underset{\text{phasor}}{\Rightarrow} \mathbf{I}_o = j\omega\, q_o$$



A **short wire** antenna has a **triangular** current distribution, since the current itself has to reach a null at the end the wires. The current can be made approximately **uniform** by adding **capacitor plates**.



The small **capacitor plate antenna** is **equivalent** to a **Hertzian dipole** and the radiated fields can also be described by using the results of the **infinitesimal antenna**. The **short wire antenna** can be described by the same results, if one uses an average current value giving the same integral of the current

$$I_o = I_{max} / 2$$

Example – A Hertzian dipole is 1.0 mm long and it operates at the frequency of 1.0 GHz, with feeding current I_o = 1.0 Ampéres. Find the total radiated power.

$$\lambda = c/f \approx 3 \times 10^8 / 10^9 = 0.3 \text{ m} = 300 \text{ mm}$$

$$\Delta z = 1 \text{ mm} \ll \lambda \quad \Rightarrow \quad \text{Hertzian dipole}$$

$$\langle P_{tot} \rangle = \frac{4\pi \eta}{3} \left(\frac{2\pi}{\lambda} \frac{I_o \Delta z}{4\pi} \right)^2 = \frac{1}{12\pi} \underbrace{120\pi}_{\eta_o} \Big(\frac{2\pi}{\underbrace{0.3}_{\lambda}} \Big)^2 \big(\underbrace{1}_{I_o} \cdot \underbrace{10^{-3}}_{\Delta z} \big)^2$$

$$= 4.39 \text{ mW}$$

For a short dipole with triangular current distribution and maximum current I_{max} = 1.0 Ampére

$$I_o = I_{max}/2 \quad \Rightarrow \quad \langle P_{tot} \rangle = 4.39/4 \approx 1.09 \text{ mW}$$

Time–dependent fields - Consider the far–field approximation

$$\vec{H}(t) = \text{Re}\left\{ \vec{\text{H}}\, e^{j\omega t} \right\} \approx \vec{i}_\varphi\, \text{Re}\left\{ \frac{j\beta \left| \vec{\text{I}} \right| \Delta z \sin\theta}{4\pi r} e^{j(\omega t - \beta r)} \right\}$$

$$\approx \vec{i}_\varphi\, \text{Re}\left\{ \frac{\beta \left| \vec{\text{I}} \right| \Delta z \sin\theta}{4\pi r} \left(j\cos(\omega t - \beta r) + j^2 \sin(\omega t - \beta r) \right) \right\}$$

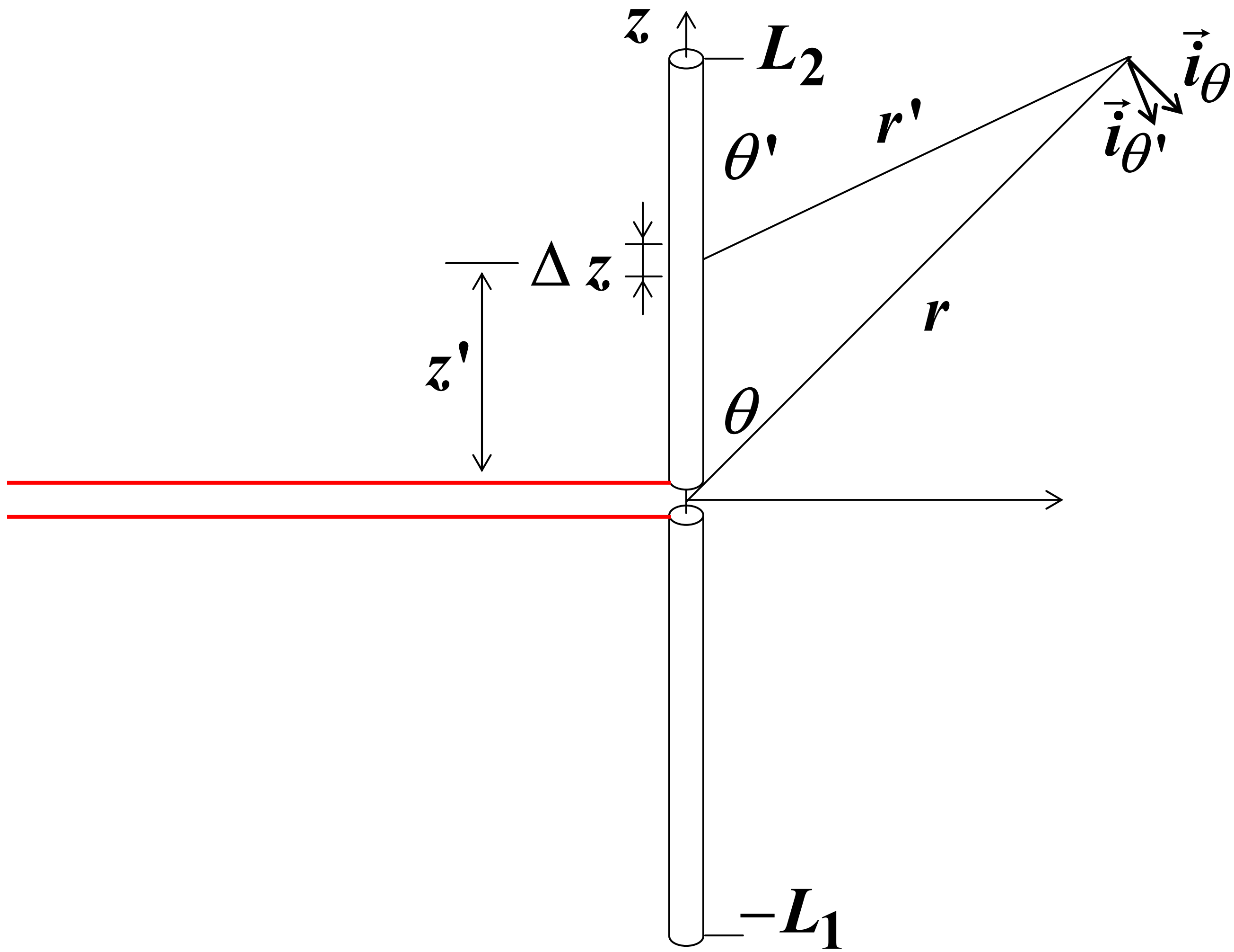
$$\approx -\vec{i}_\varphi \frac{\beta \left| \vec{\text{I}} \right| \Delta z \sin\theta}{4\pi r} \sin(\omega t - \beta r)$$

$$\vec{E}(t) = \text{Re}\left\{ \vec{\text{E}}\, e^{j\omega t} \right\}$$

$$\approx -\vec{i}_\theta\, \eta \frac{\beta \left| \vec{\text{I}} \right| \Delta z \sin\theta}{4\pi r} \sin(\omega t - \beta r)$$

Linear Antennas

Consider a dipole with wires of length comparable to the wavelength.



Because of its length, the current flowing in the antenna wire is a function of the coordinate z. To evaluate the far–field at an observation point, we divide the antenna into segments which can be considered as elementary infinitesimal antennas.

The electric field radiated by each element , in the far–field approximation, is

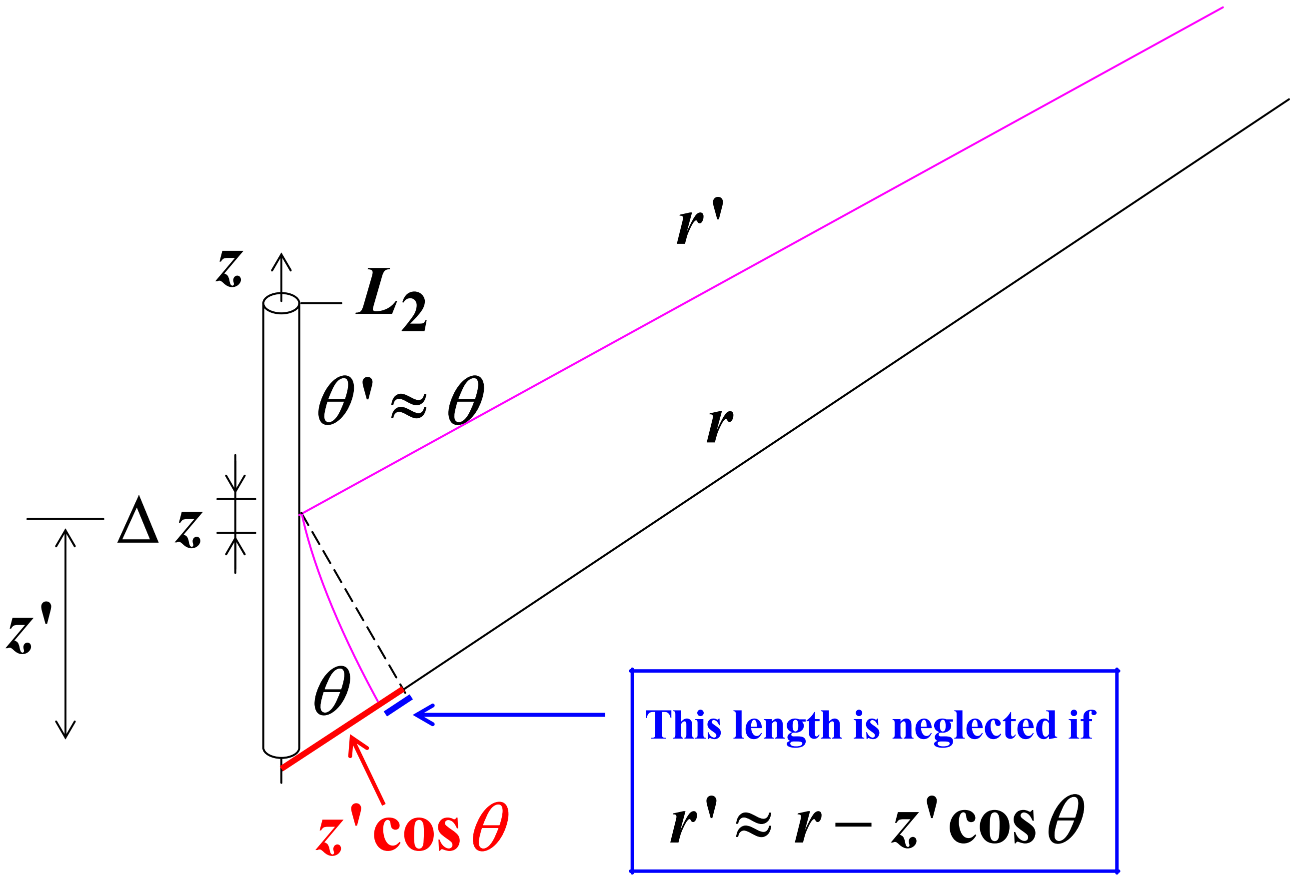
$$\Delta E' = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta \left| \vec{\mathrm{I}} \right| \Delta z \, e^{-j\beta r'}}{4\pi r'} \sin\theta'$$

In far–field conditions we can use these additional approximations

$$\theta \approx \theta'$$

$$r' \approx r - z' \cos\theta$$

The lines r and r' are nearly parallel under these assumptions.



The electric field contributions due to each infinitesimal segment becomes

$$\Delta E' = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta \left| \vec{I} \right| \Delta z \, e^{-j\beta r} \overbrace{e^{j\beta z' \cos\theta}}^{\text{you cannot neglect here}}}{4\pi r \underbrace{- 4\pi z' \cos\theta}_{\text{you can neglect here}}} \sin\theta$$

The total fields are obtained by integration of all the contributions

$$\vec{E} = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta \, e^{-j\beta r}}{4\pi r} \sin\theta \cdot \int_{-L_1}^{L_2} I(z) \, e^{j\beta z \cos\theta} \, dz$$

$$\vec{H} = \vec{i}_\varphi \frac{j\beta \, e^{-j\beta r}}{4\pi r} \sin\theta \cdot \int_{-L_1}^{L_2} I(z) \, e^{j\beta z \cos\theta} \, dz$$

Short Dipole

Consider a short symmetric dipole comprising two wires, each of length $L \ll \lambda$. Assume a triangular distribution of the phasor current on the wires

$$\mathrm{I}(z) = \begin{cases} I_{max}\left(1 - z/L\right) & z \geq 0 \\ I_{max}\left(1 + z/L\right) & z < 0 \end{cases}$$

The integral in the field expressions becomes

$$\int_{-L}^{L} \mathrm{I}(z)\ \underbrace{e^{j\beta z\cos\theta}}_{\approx 1}\ dz \approx \int_{-L}^{L} \mathrm{I}(z)\ dz = \frac{2L}{2} I_{max}$$

since $\max|\beta z| = \beta \cdot L = \dfrac{2\pi}{\lambda} L \ll 1$ for a short dipole

$$\Rightarrow \quad e^{j\beta z\cos\theta} \approx 1$$

The final expression for far–fields of the short dipole are similar to the expressions for the Hertzian dipole where the average of the triangular current distribution is used

$$\vec{\mathbf{E}} = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta\, e^{-j\beta r}}{4\pi r} \sin\theta \cdot \overbrace{2L}^{\Delta z} \cdot \underbrace{\frac{I_{\max}}{2}}_{\text{average current}}$$

$$= \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta\, I_{\max} L\, e^{-j\beta r}}{4\pi r} \sin\theta$$

$$\vec{\mathbf{H}} = \vec{i}_\varphi \frac{j\beta\, I_{\max} L\, e^{-j\beta r}}{4\pi r} \sin\theta$$

Half–wavelength dipole

Consider a symmetric linear antenna with total length λ/2 and assume a current phasor distribution on the wires which is approximately sinusoidal

$$\mathrm{I}(z) = I_{\max} \cos(\beta\, z)$$

The integral in the field expressions is

$$\int_{-\lambda/4}^{\lambda/4} I_{\max} \cos(\beta\, z)\, e^{j\beta z \cos\theta}\, dz = \frac{2 I_{\max}}{\beta \sin^2\theta} \cos\left(\frac{\pi \cos\theta}{2}\right)$$

We obtain the far–field expressions

$$\vec{\mathbf{E}} = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\, e^{-j\beta r}}{2\pi\, r} \frac{I_{\max}}{\sin\theta} \cos\left(\frac{\pi \cos\theta}{2}\right)$$

$$\vec{\mathbf{H}} = \vec{i}_\varphi \frac{j\, e^{-j\beta r}}{2\pi\, r} \frac{I_{\max}}{\sin\theta} \cos\left(\frac{\pi \cos\theta}{2}\right)$$

and the time–average Poynting vector

$$\left\langle \vec{P}(t) \right\rangle = \vec{i}_r \sqrt{\frac{\mu}{\varepsilon}} \frac{I_{\max}^2}{8\pi^2\, r^2 \sin^2\theta} \cos^2\left(\frac{\pi \cos\theta}{2}\right)$$

The total radiated power is obtained after integration of the time–average Poynting vector

$$P_{tot} = \frac{1}{2} I_{\max}^2 \sqrt{\frac{\mu}{\varepsilon}} \frac{1}{4\pi} \underbrace{\int_0^{2\pi} \left(\frac{1 - \cos(u)}{u} \right) du}_{\approx 2.4376}$$

$$= \frac{1}{2} I_{\max}^2 \underbrace{\sqrt{\frac{\mu}{\varepsilon}} \cdot 0.193978}_{R_{eq}}$$

The integral above cannot be solved analytically, but the value is found numerically or from published tables. The equivalent resistance of the half–wave dipole antenna in air is then

$$R_{eq}(\lambda/2) = \sqrt{\frac{\mu}{\varepsilon}} \cdot 0.193978 \approx 73.07\ \Omega$$

The direction of maximum radiation strength is obtained again for polar angle θ=90° ande we obtain the directivity

$$D = \frac{\left|\left\langle \vec{P}(t, r, 90°)\right\rangle\right|}{P_{tot} \big/ 4\pi r^2} = \frac{\sqrt{\dfrac{\mu}{\varepsilon}}\, \dfrac{I_{max}^2}{8\pi^2 r^2}}{\dfrac{1}{8\pi^2 r^2} I_{max}^2 \sqrt{\dfrac{\mu}{\varepsilon}} \cdot 2.4376} \approx 1.641$$

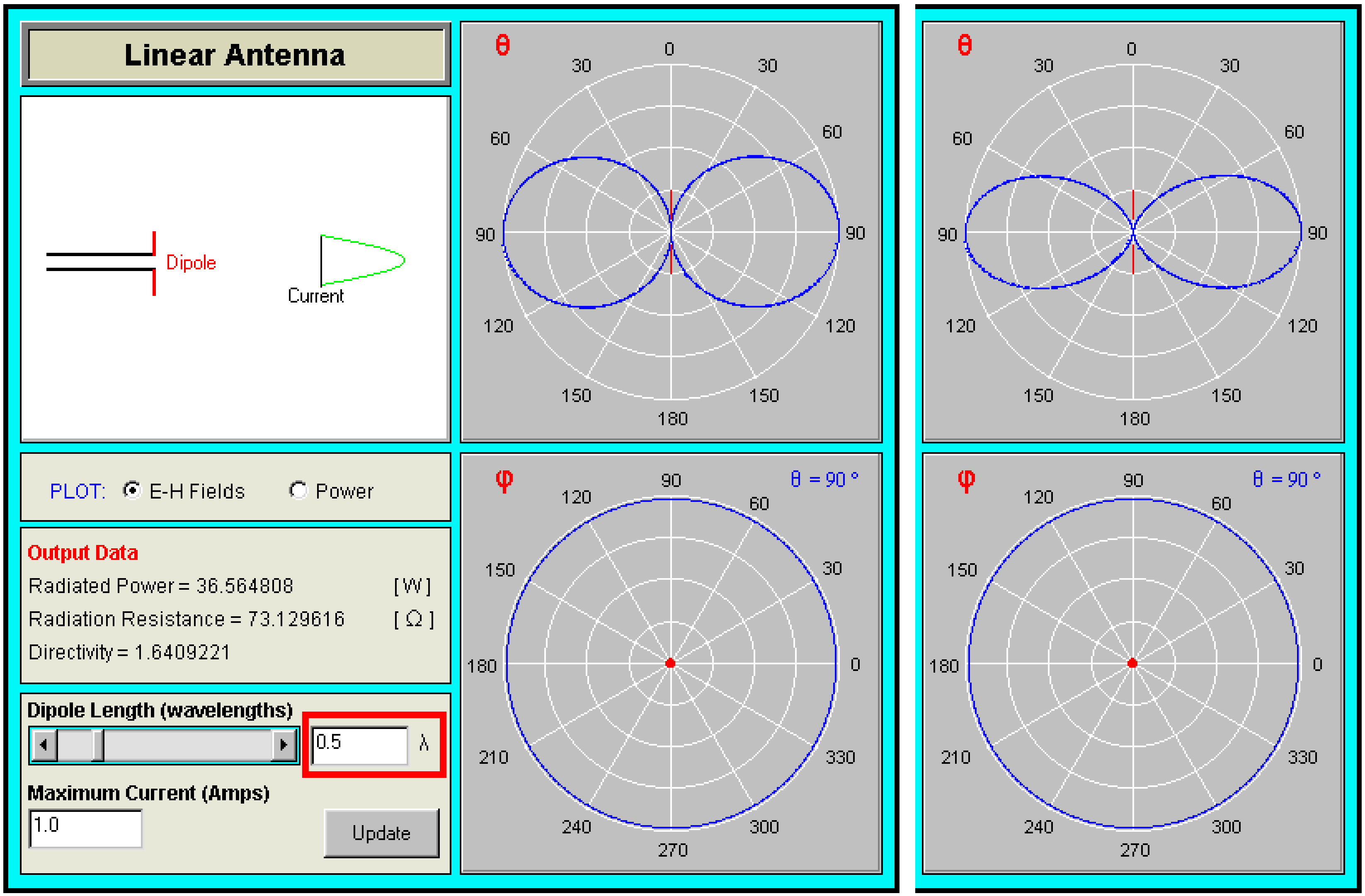
The directivity of the half–wavelength dipole is marginally better than the directivity for a Hertzian dipole ($D = 1.5$).

The real improvement is in the much larger radiation resistance, which is now comparable to the characteristic impedance of typical transmission line.

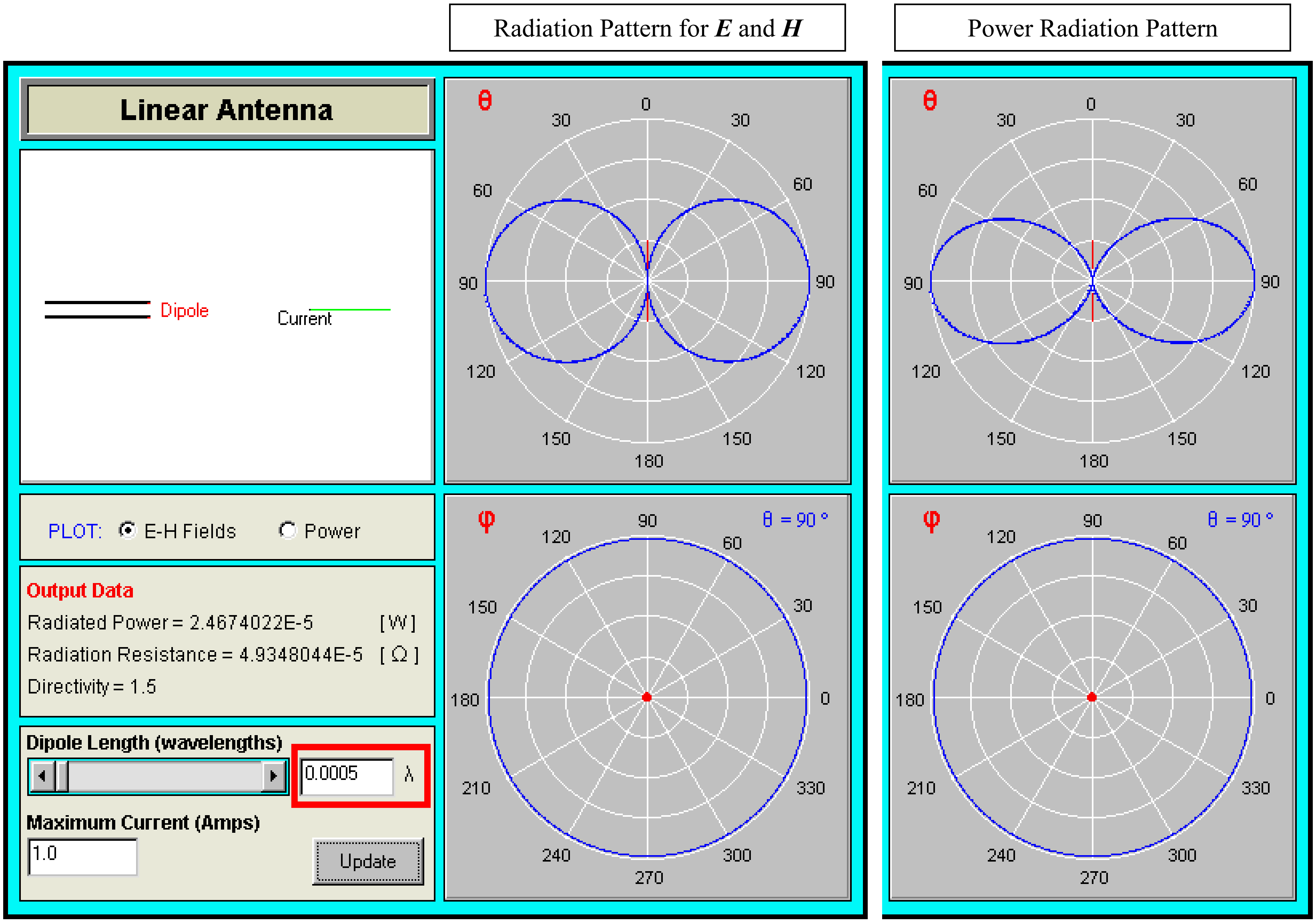
From the linear antenna applet

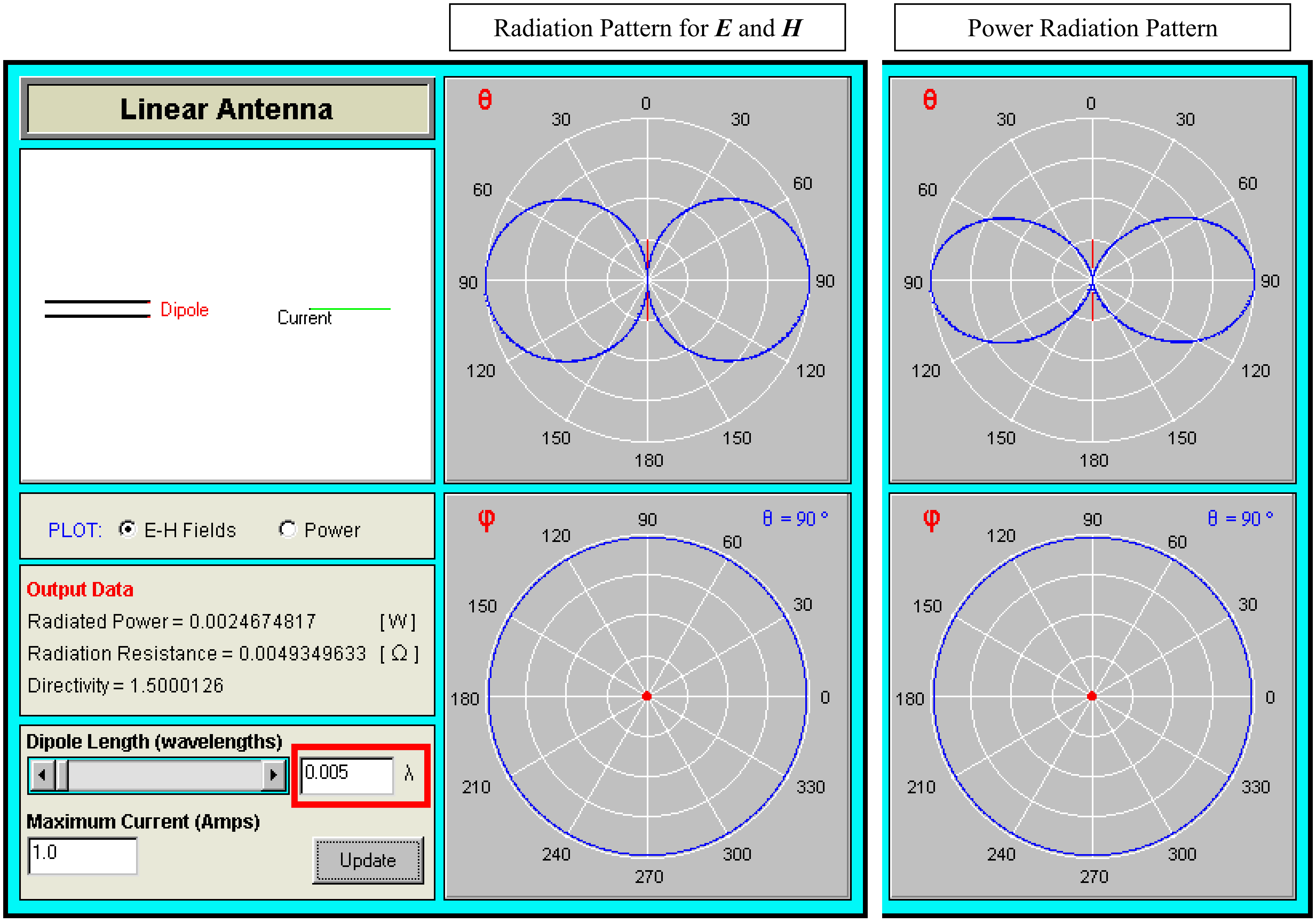
Radiation Pattern for ***E*** and ***H***

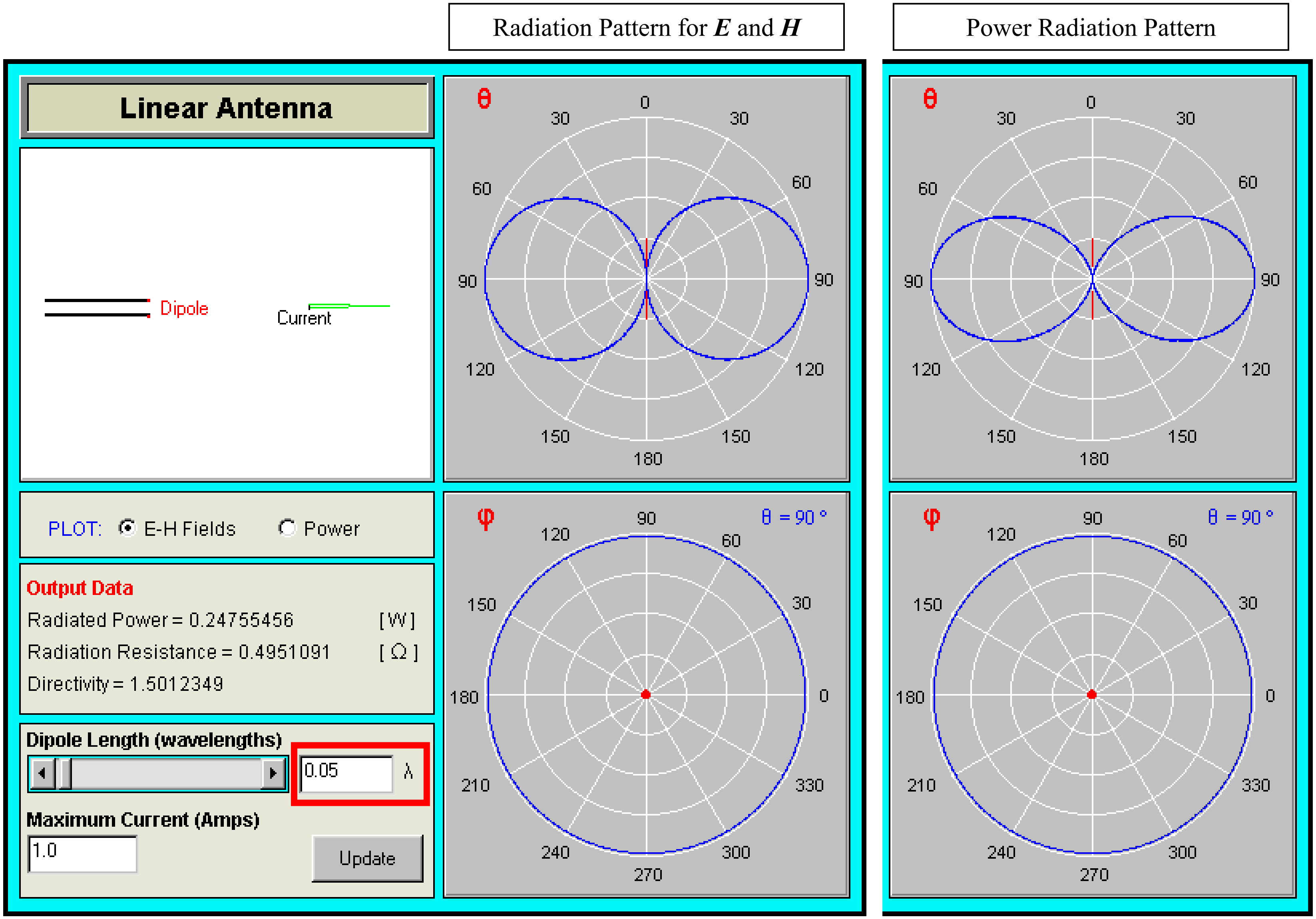
Power Radiation Pattern



For short dipoles of length 0.0005 λ to 0.05 λ




Radiation Pattern for *E* and *H*
Power Radiation Pattern
Linear Antenna
Dipole
Current
PLOT: E-H Fields Power
Output Data
Radiated Power = 0.0024674817 [W]
Radiation Resistance = 0.0049349633 [Ω]
Directivity = 1.5000126
Dipole Length (wavelengths)
0.005 λ
Maximum Current (Amps)
1.0
Update
θ = 90 °



Radiation Pattern for E and H
Power Radiation Pattern
Linear Antenna
Dipole
Current
PLOT: E-H Fields Power
Output Data
Radiated Power = 0.24755456 [W]
Radiation Resistance = 0.4951091 [Ω]
Directivity = 1.5012349
Dipole Length (wavelengths)
0.05 λ
Maximum Current (Amps)
1.0
Update
θ = 90 °

For general symmetric linear antennas with two wires of length L, it is convenient to express the current distribution on the wires as

$$\mathbf{I}(z) = I_{\max} \sin\{\beta(L - |z|)\}$$

The integral in the field expressions is now

$$\int_{-L}^{L} I_{\max} \sin[\beta(L - |z|)]\, e^{j\beta z \cos\theta}\, dz =$$

$$= \frac{2 I_{\max}}{\beta \sin^2 \theta} \{\cos(\beta L \cos\theta) - \cos(\beta L)\}$$

The field expressions become

$$\vec{\mathrm{E}} = \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\beta\, e^{-j\beta r}}{4\pi\, r} \sin\theta \cdot \int_{-L_1}^{L_2} \mathrm{I}(z)\, e^{j\beta z\cos\theta}\, dz$$

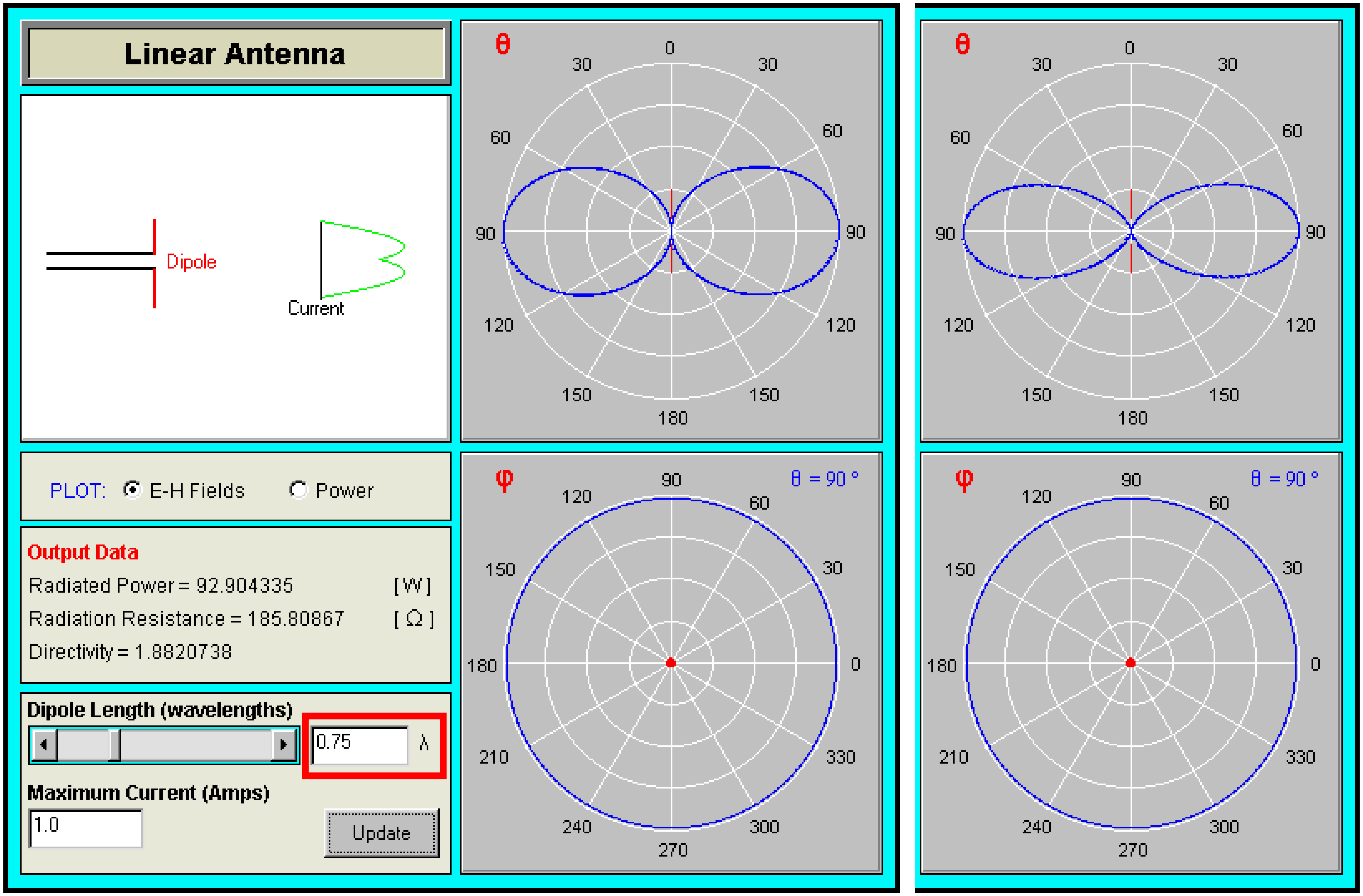
$$= \vec{i}_\theta \sqrt{\frac{\mu}{\varepsilon}} \frac{j\, I_{\max}\, e^{-j\beta r}}{2\pi\, r \sin\theta} \left\{ \cos\left(\beta L \cos\theta\right) - \cos\left(\beta L\right) \right\}$$

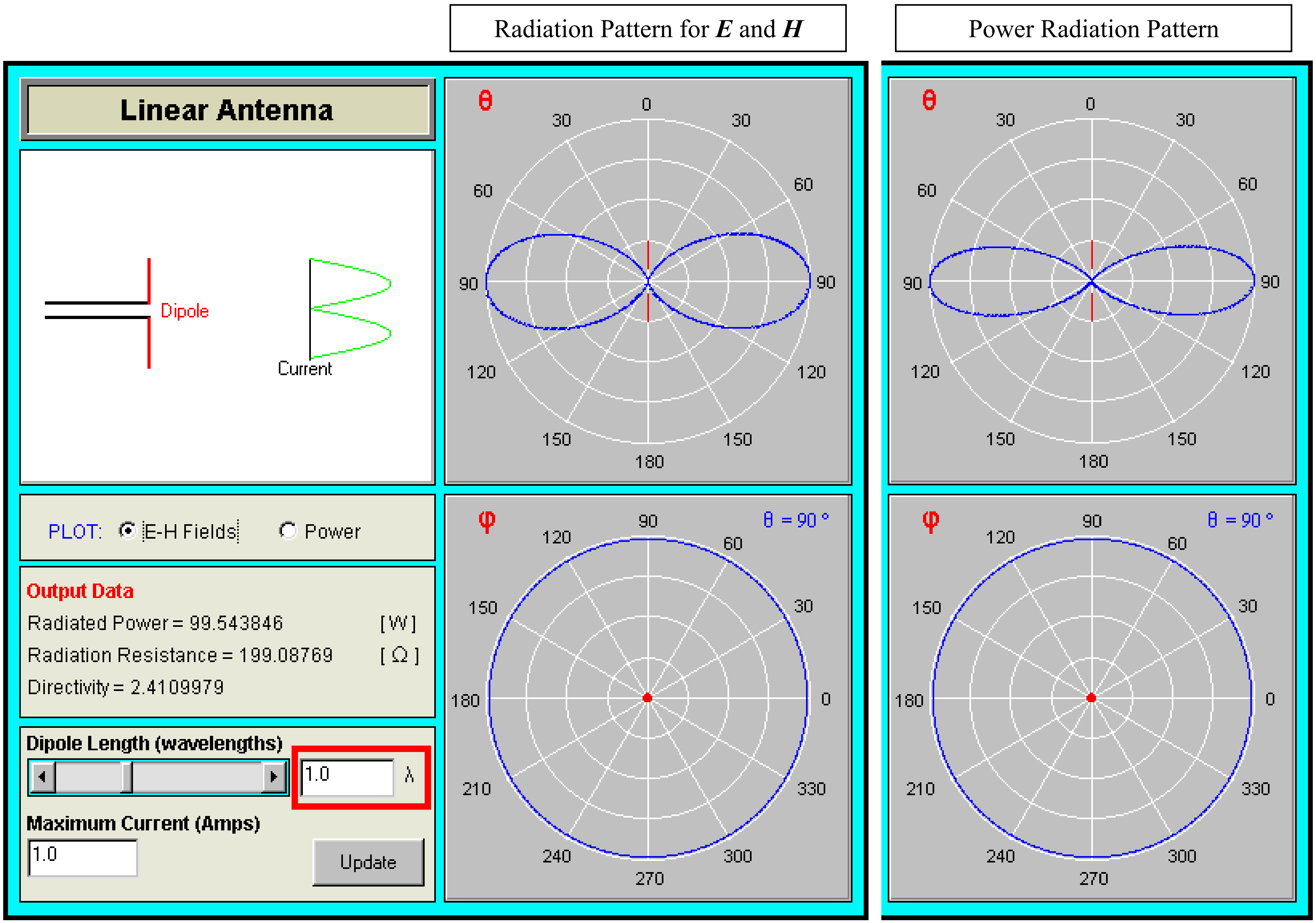
$$\vec{\mathrm{H}} = \vec{i}_\varphi \frac{j\beta\, e^{-j\beta r}}{4\pi\, r} \sin\theta \cdot \int_{-L_1}^{L_2} \mathrm{I}(z)\, e^{j\beta z\cos\theta}\, dz$$

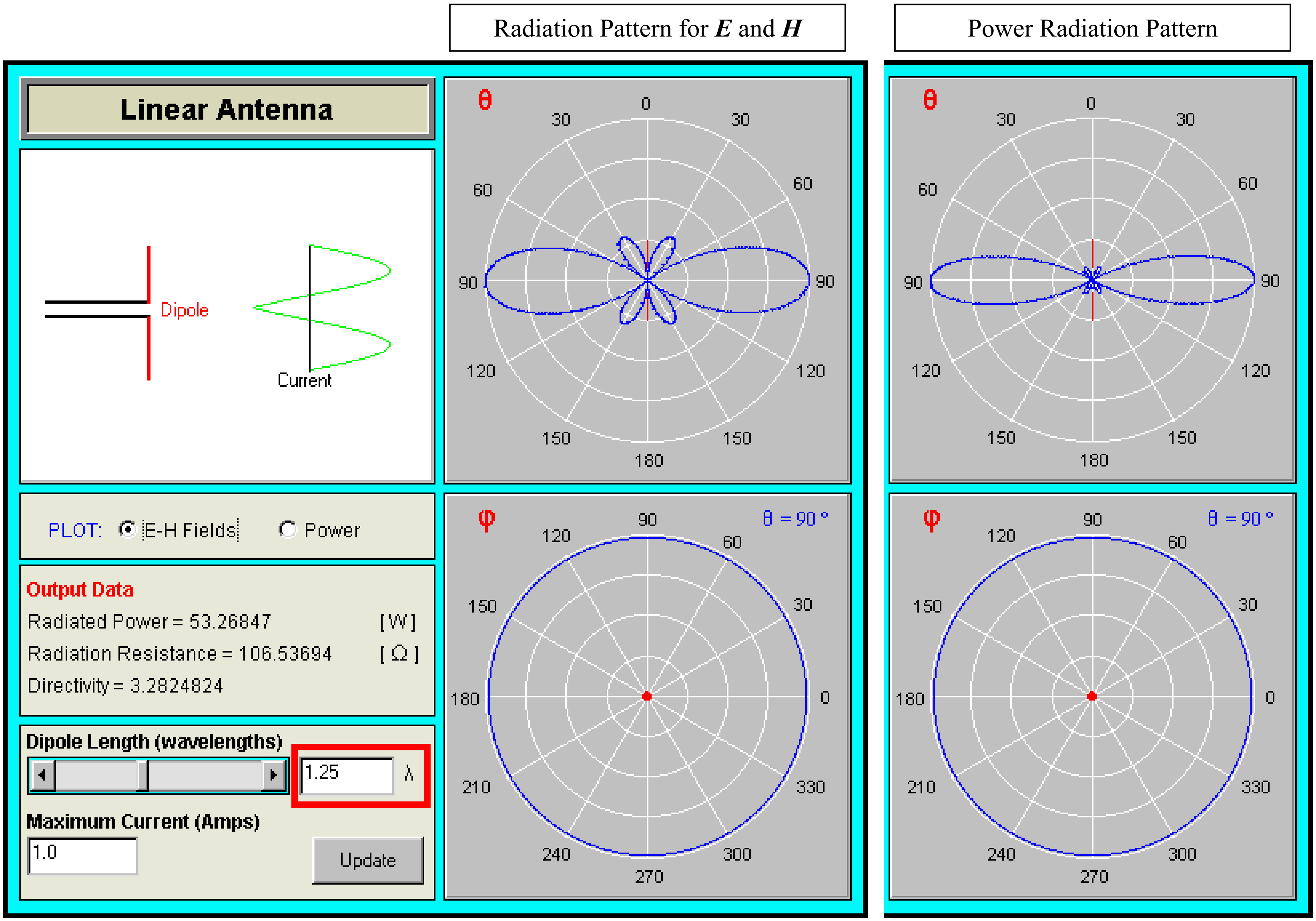
$$= \vec{i}_\varphi \frac{j\, I_{\max}\, e^{-j\beta r}}{2\pi\, r \sin\theta} \left\{ \cos\left(\beta L \cos\theta\right) - \cos\left(\beta L\right) \right\}$$

Examples of long wire antennas

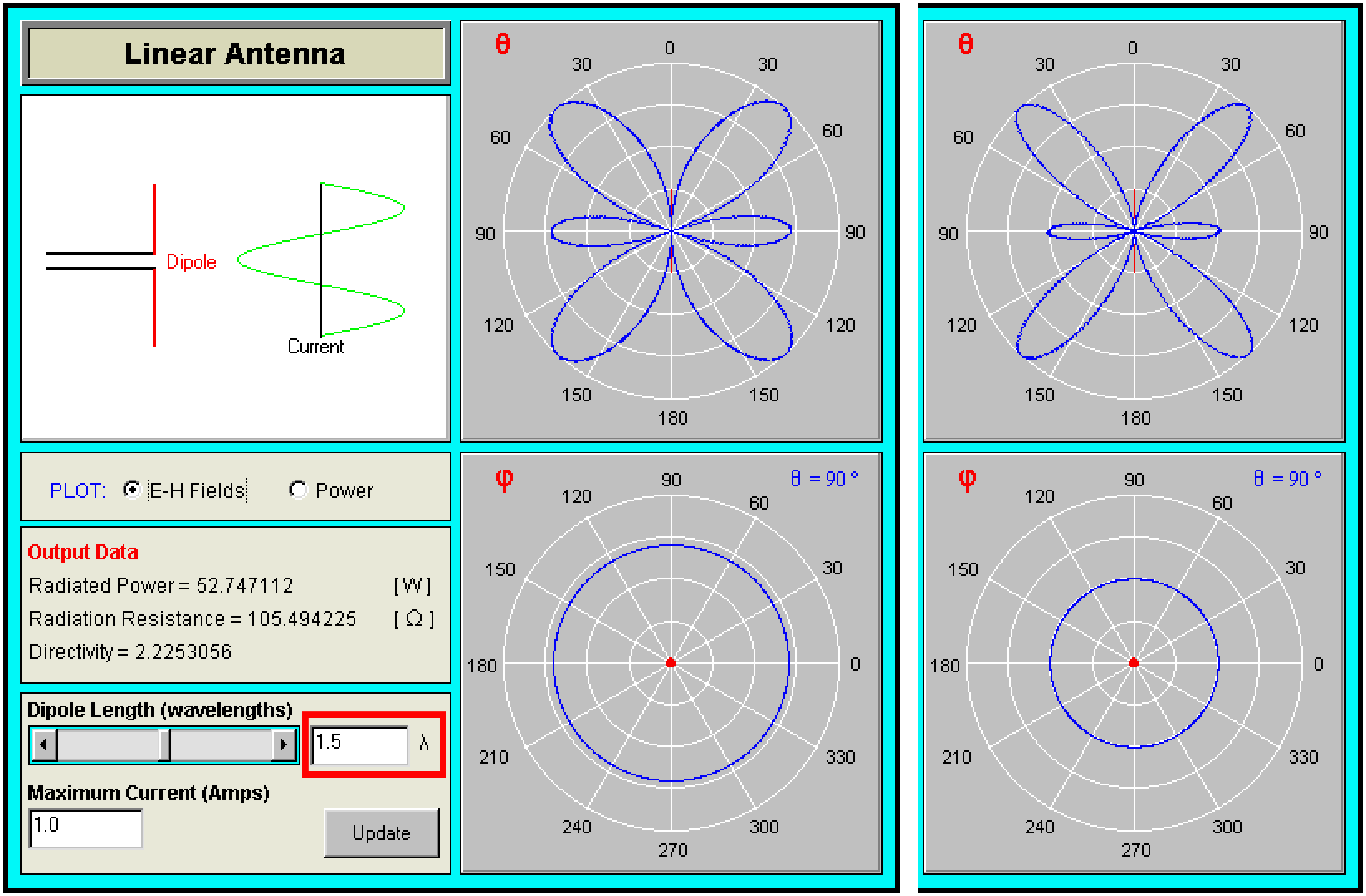
Radiation Pattern for ***E*** and ***H*** | Power Radiation Pattern



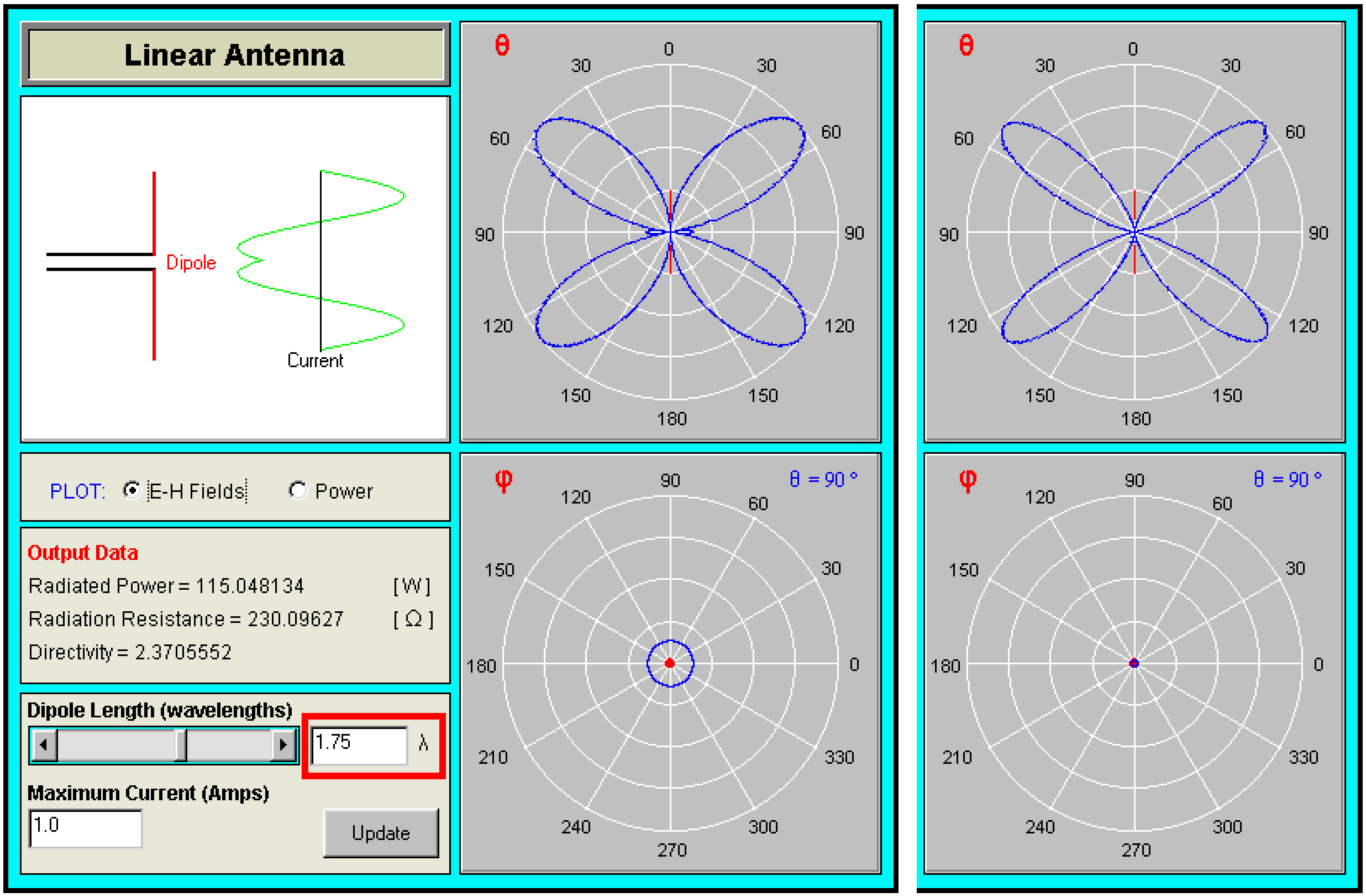

Radiation Pattern for E and H
Power Radiation Pattern
Linear Antenna
Dipole
Current
PLOT: E-H Fields Power
Output Data
Radiated Power = 99.543846 [W]
Radiation Resistance = 199.08769 [Ω]
Directivity = 2.4109979
Dipole Length (wavelengths)
1.0 λ
Maximum Current (Amps)
1.0
Update
θ = 90 °

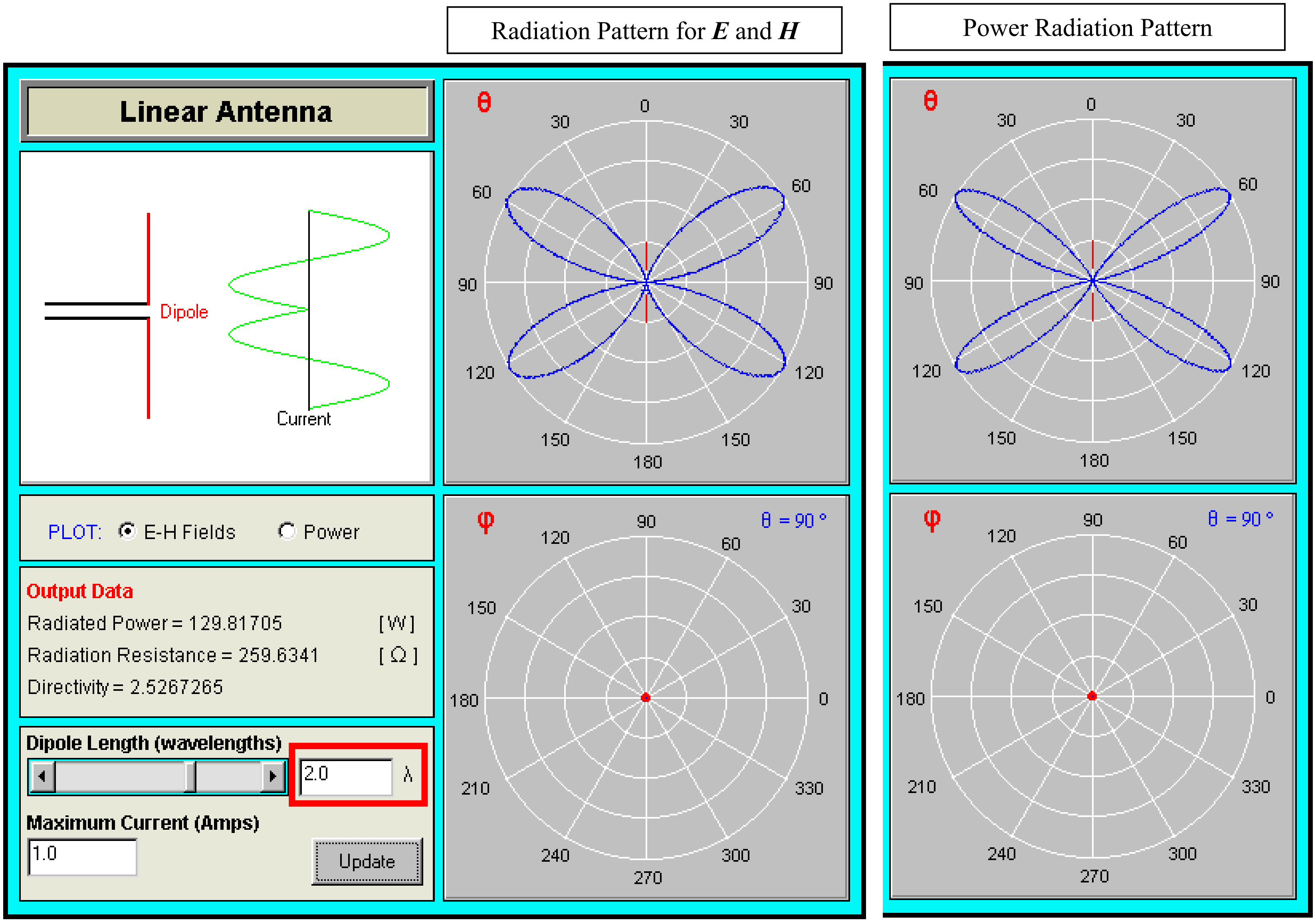

Radiation Pattern for E and H
Power Radiation Pattern
Linear Antenna
Dipole
Current
θ
0
30
60
90
120
150
180
φ
θ = 90 °
270
240
210
330
300
PLOT: E-H Fields Power
Output Data
Radiated Power = 53.26847 [W]
Radiation Resistance = 106.53694 [Ω]
Directivity = 3.2824824
Dipole Length (wavelengths)
1.25 λ
Maximum Current (Amps)
1.0
Update

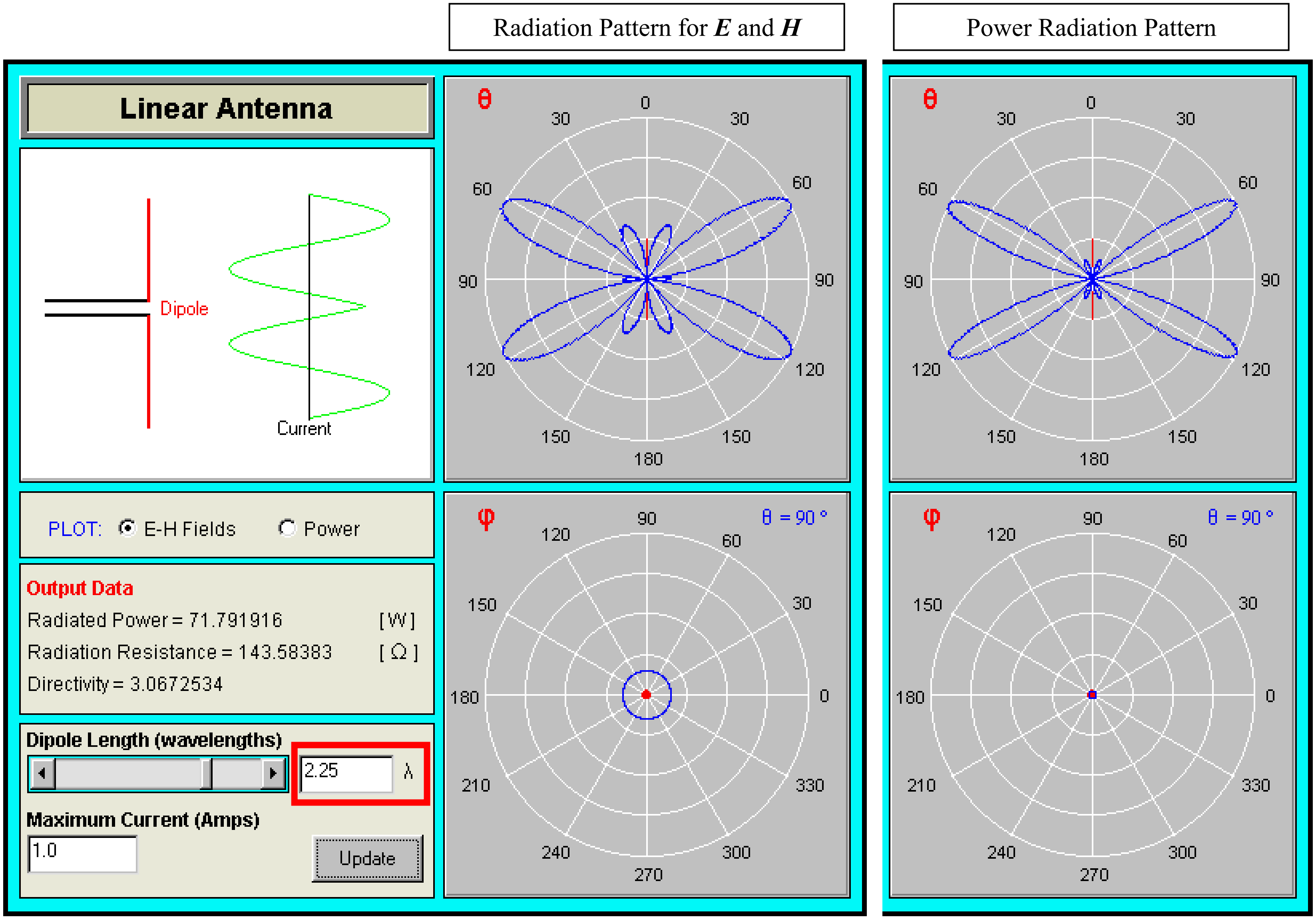
Radiation Pattern for ***E*** and ***H*** | Power Radiation Pattern



Radiation Pattern for ***E*** and ***H*** | Power Radiation Pattern

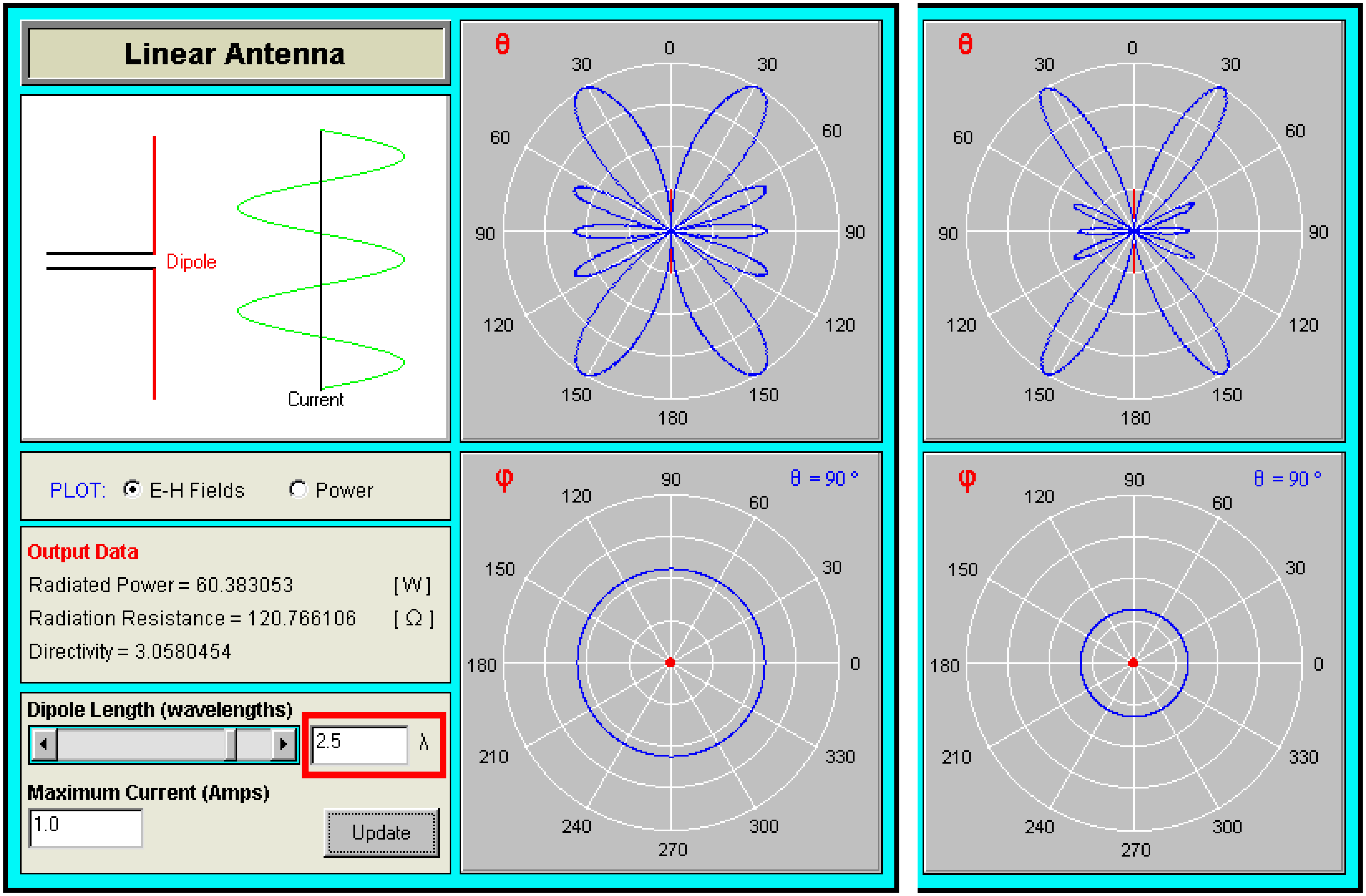



Radiation Pattern for E and H
Power Radiation Pattern
Linear Antenna
Dipole
Current
PLOT: E-H Fields Power
Output Data
Radiated Power = 129.81705 [W]
Radiation Resistance = 259.6341 [Ω]
Directivity = 2.5267265
Dipole Length (wavelengths)
2.0 λ
Maximum Current (Amps)
1.0
Update
θ = 90 °

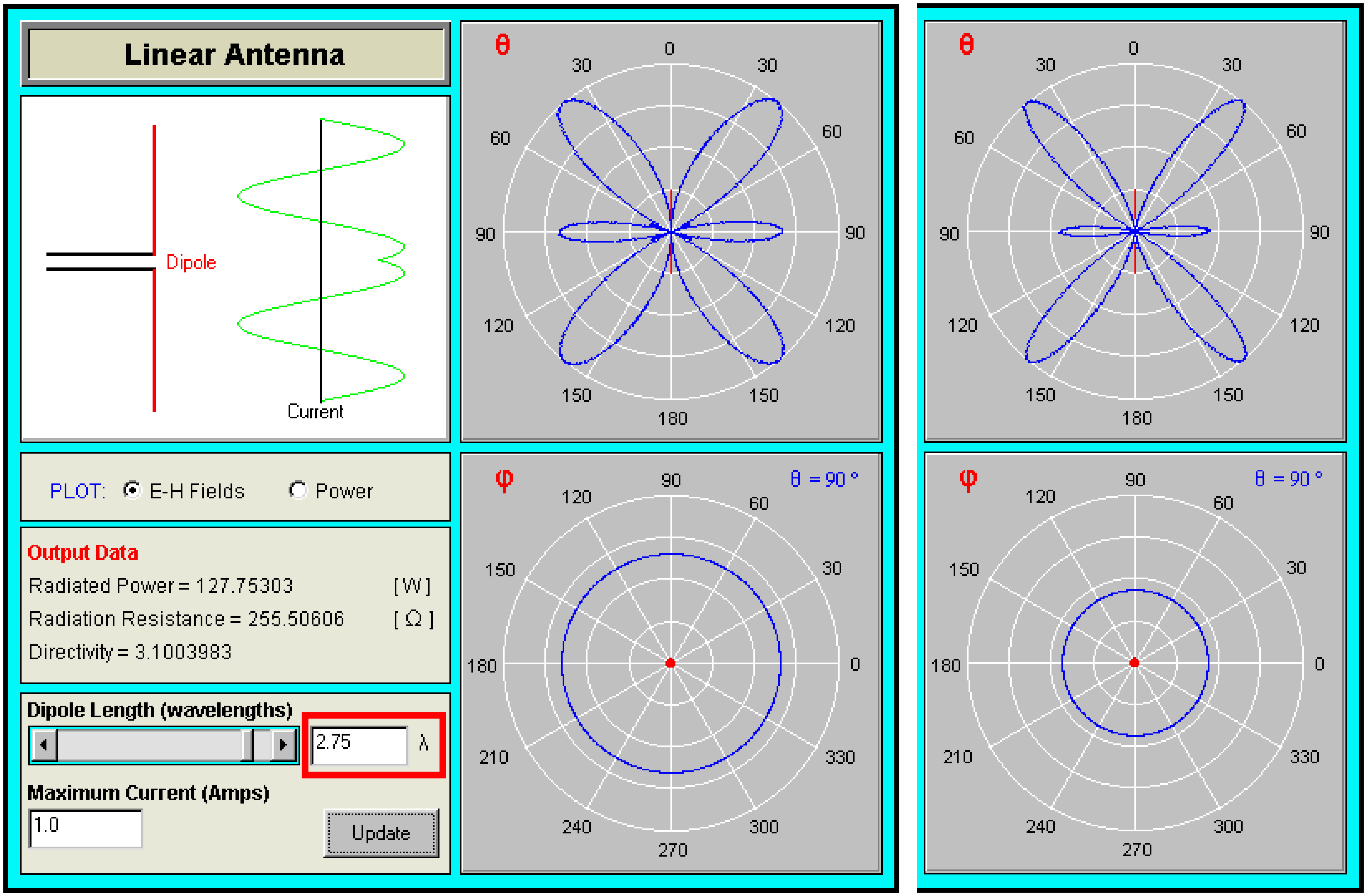


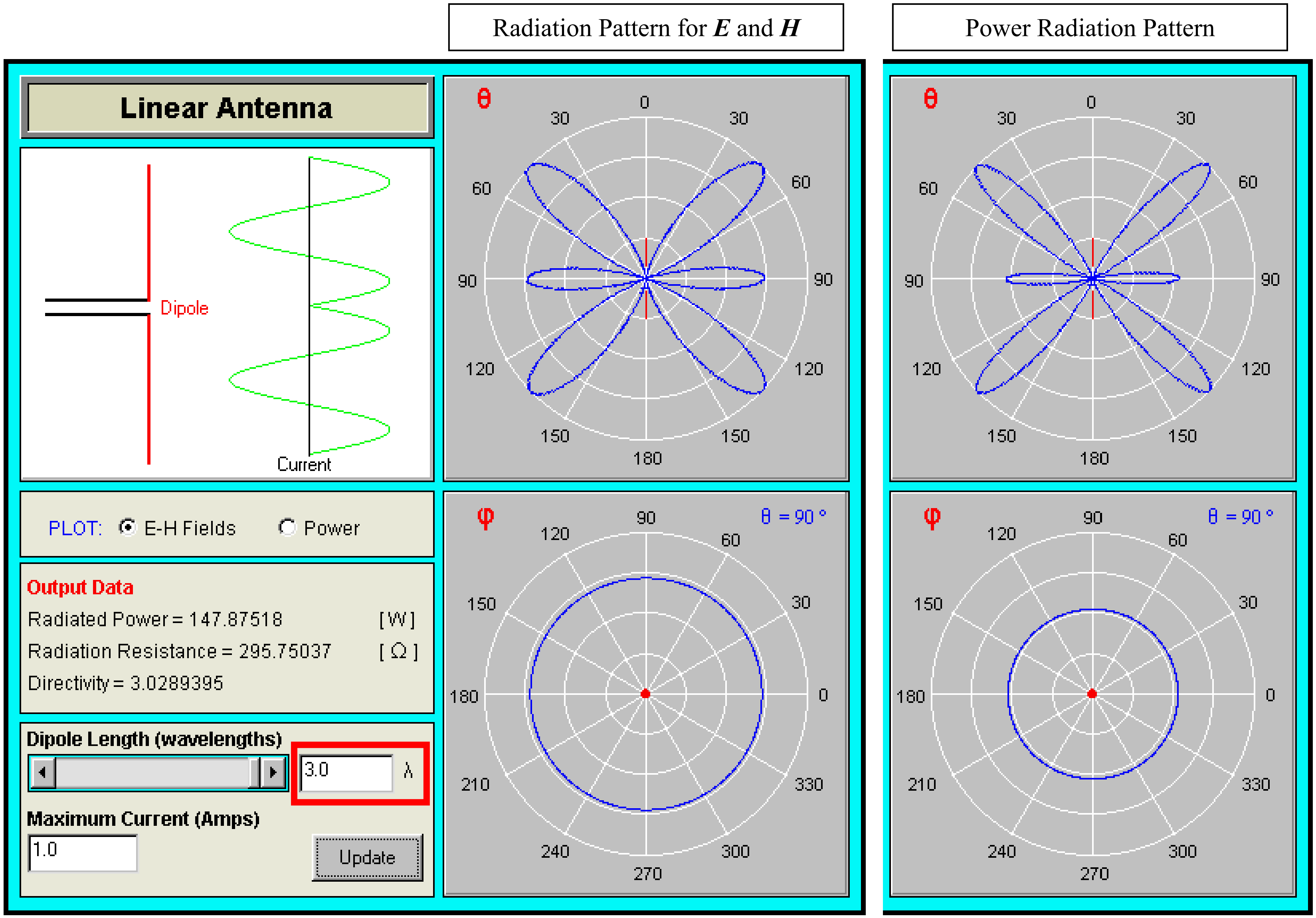
Radiation Pattern for *E* and *H*
Power Radiation Pattern
Linear Antenna
Dipole
Current
PLOT: E-H Fields Power
Output Data
Radiated Power = 71.791916 [W]
Radiation Resistance = 143.58383 [Ω]
Directivity = 3.0672534
Dipole Length (wavelengths)
2.25 λ
Maximum Current (Amps)
1.0
Update
θ
φ
θ = 90 °

Radiation Pattern for ***E*** and ***H*** | Power Radiation Pattern



Radiation Pattern for ***E*** and ***H*** — Power Radiation Pattern




Radiation Pattern for E and H
Power Radiation Pattern
Linear Antenna
Dipole
Current
PLOT: E-H Fields Power
Output Data
Radiated Power = 147.87518 [W]
Radiation Resistance = 295.75037 [Ω]
Directivity = 3.0289395
Dipole Length (wavelengths)
3.0 λ
Maximum Current (Amps)
1.0
Update
θ
φ
θ = 90 °