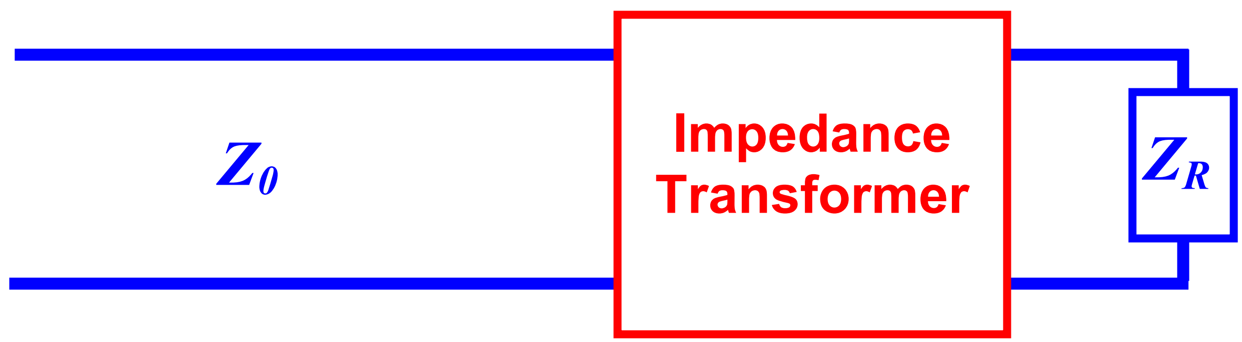
Impedance Matching

A number of techniques can be used to eliminate reflections when the characteristic impedance of the line and the load impedance are mismatched.

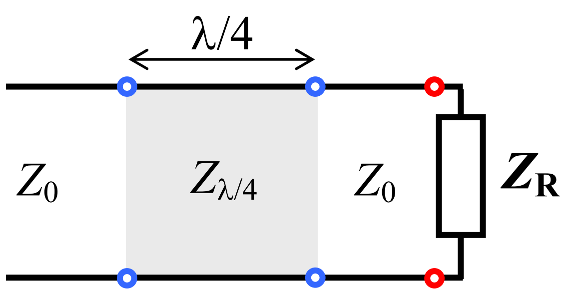
Impedance matching techniques can be designed to be effective for a specific frequency of operation (narrow band techniques) or for a given frequency spectrum (broadband techniques).

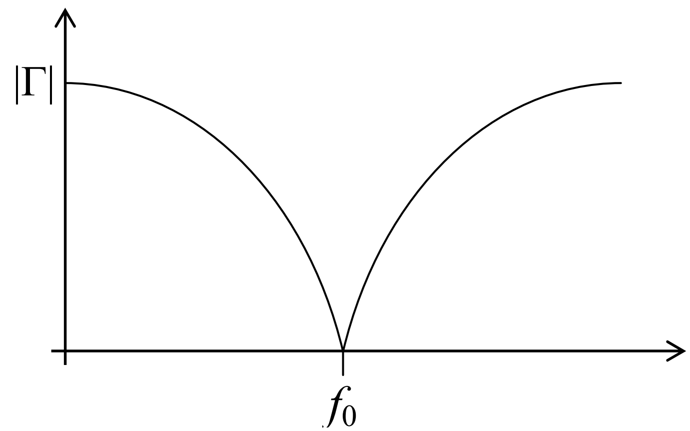
A common method of impedance matching involves the insertion of an impedance transformer between line and load



An impedance transformer may be realized by inserting a section of a different transmission line with appropriate characteristic impedance. A widely used approach realizes the transformer with a line of length $\lambda/4$.

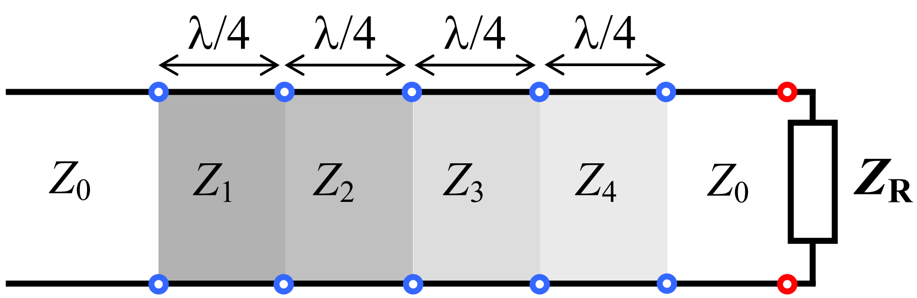
The quarter-wavelength transformer provides narrow-band impedance matching. The design goal is to obtain zero reflection coefficient exactly at the frequency of operation.

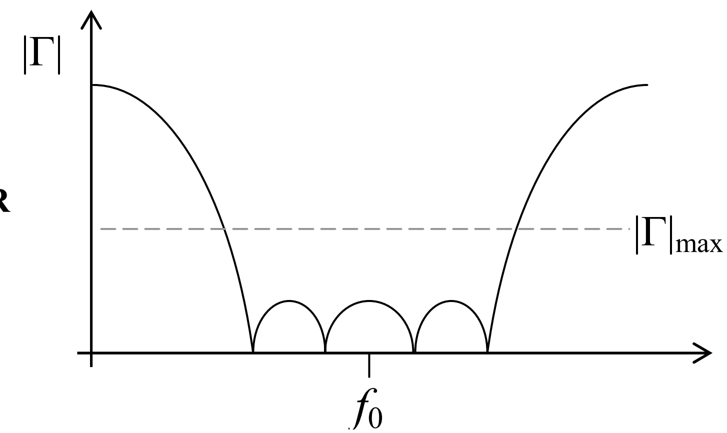




The length of the transformer is fixed at $\lambda/4$ for design convenience, but is also possible to realize generalized transformer lines for which the length of the transformer is a design outcome.

A broadband design may be obtained by a cascade of $\lambda/4$ line sections of gradually varying characteristic impedance.

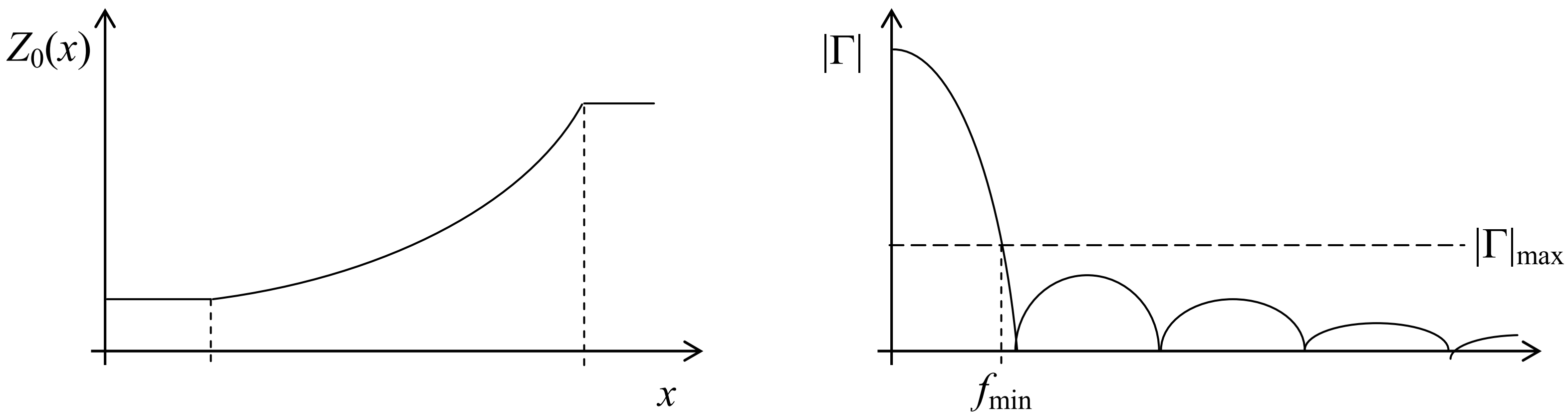




It is not possible to obtain exactly zero reflection coefficient for all frequencies in the desired band.

Therefore, available design approaches specify a maximum reflection coefficient (or maximum VSWR) which can be tolerated in the frequency band of operation.

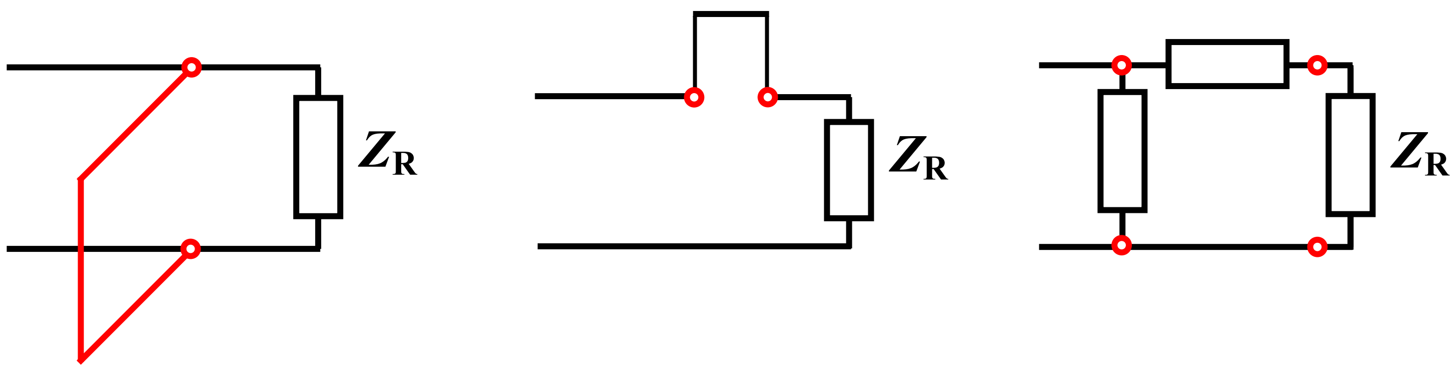
Another broadband matching approach may use a tapered line transformer with continously varying characteristic impedance along its length. In this case, the design obtains reflection coefficients lower than a specified tolerance at frequencies exceeding a minimum value.



Various taper designs are available, including linear, exponential, and raised-cosine impedance profiles. An optimal design (due to Klopfenstein) involves discontinuity of the impedance at the transformer ends.

Another narrow-band approach involves the insertion of a shunt imaginary admittance on the line. Often, the admittance is realized with a section (or stub) of transmission line and the technique is commonly known as stub matching. The end of the stub line is short-circuited or open-circuited, in order to realize an imaginary admittance. Designs are also available for two or three shunt admittances placed at specified locations on the line.

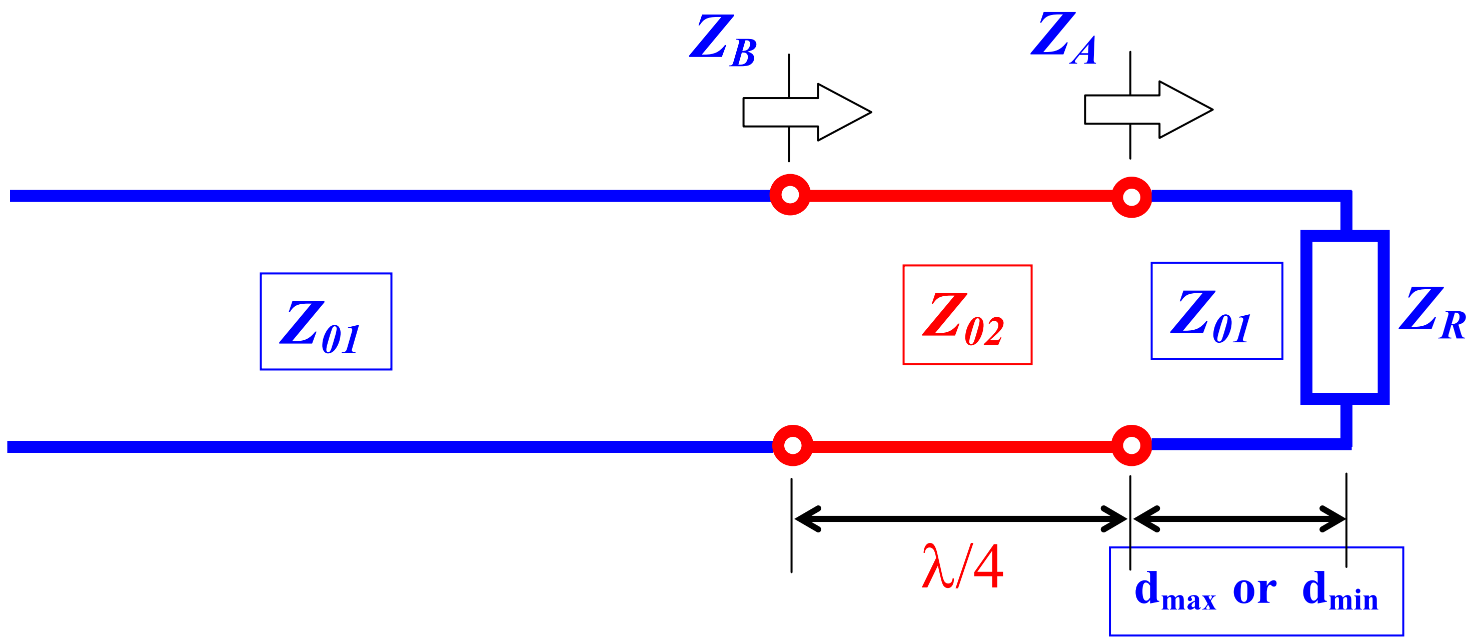
Other narrow-band examples involve the insertion of a series impedance (stub) along the line, and the insertion of a series and a shunt element in L-configuration.



The theory for several basic narrow-band matching techniques is detailed in the following. Note that the effect of loss in the transmission lines is always neglected.

Matching I: Impedance Transformers

- **Quarter Wavelength Transformer – A simple narrow band impedance transformer consists of a transmission line section of length** $\lambda/4$



The impedance transformer is positioned so that it is connected to a real impedance Z_A**. This is always possible if a location of maximum or minimum voltage standing wave pattern is selected.**

Consider a general load impedance with its corresponding load reflection coefficient

$$Z_R = R_R + jX_R \quad ; \qquad \Gamma_R = \frac{Z_R - Z_{01}}{Z_R + Z_{01}} = |\Gamma_R| \exp(j\phi)$$

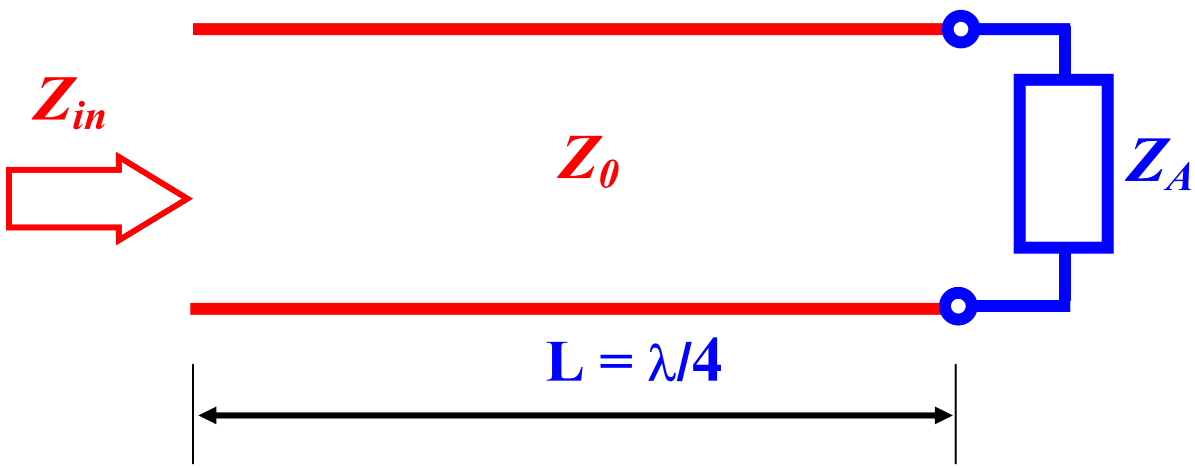
If the transformer is inserted at a location of voltage maximum $\mathbf{d_{max}}$

$$Z_A = Z_{01} \frac{1 + \Gamma(\mathrm{d})}{1 - \Gamma(\mathrm{d})} = Z_{01} \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|}$$

If it is inserted instead at a location of voltage minimum $\mathbf{d_{min}}$

$$Z_A = Z_{01} \frac{1 + \Gamma(\mathrm{d})}{1 - \Gamma(\mathrm{d})} = Z_{01} \frac{1 - |\Gamma_R|}{1 + |\Gamma_R|}$$

Consider now the input impedance of a line of length $\lambda/4$



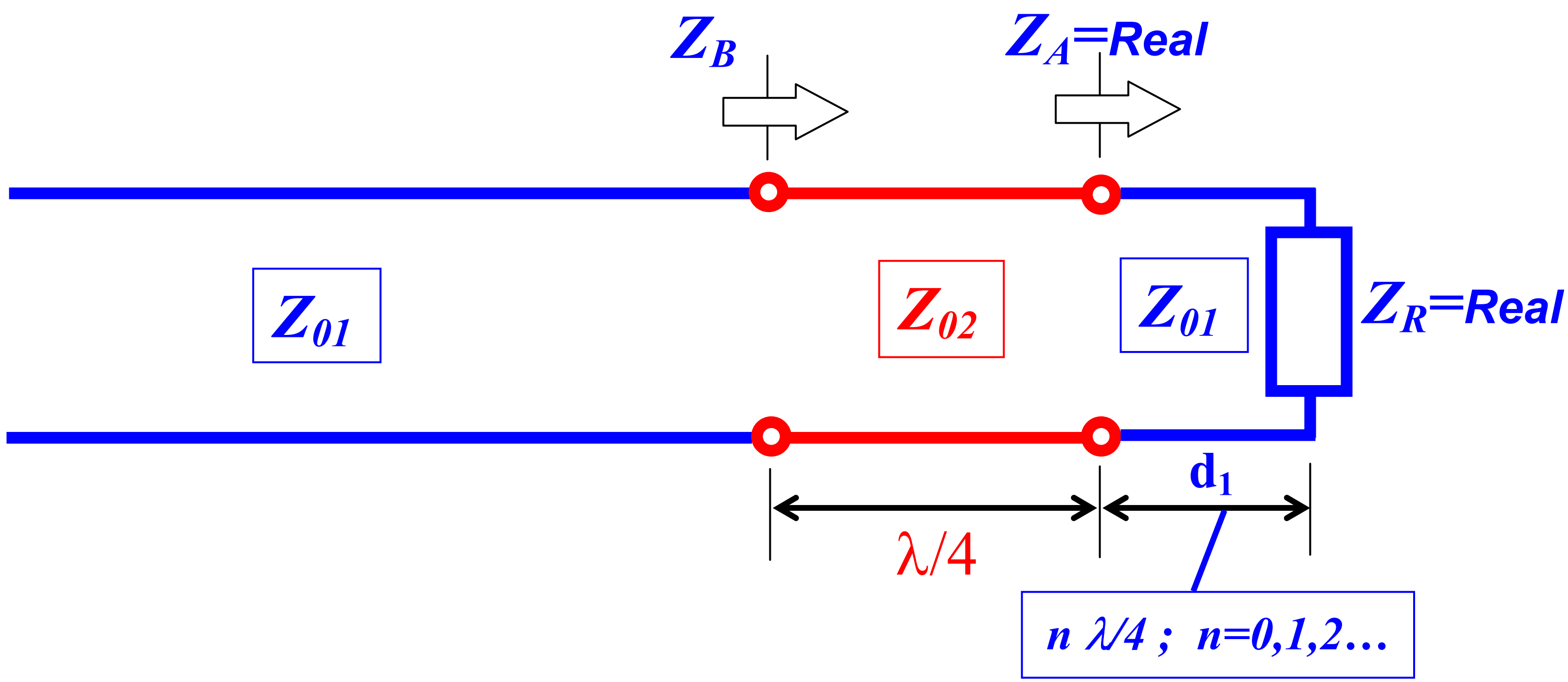
Since:

$$Z_A = Z_{01} \frac{1+\Gamma(d)}{1-\Gamma(d)} = Z_{01} \frac{1-|\Gamma_R|}{1+|\Gamma_R|}$$

we have

$$Z_{in} = \lim_{\tan(\beta L)\to\infty} Z_0 \frac{Z_A + jZ_0 \tan(\beta L)}{jZ_A \tan(\beta L) + Z_0} \rightarrow \frac{Z_0^2}{Z_A}$$

Note that if the load is real, the voltage standing wave pattern at the load is maximum when $Z_R > Z_{01}$ or minimum when $Z_R < Z_{01}$. The transformer can be connected directly at the load location or at a distance from the load corresponding to a multiple of $\lambda/4$.



If the load impedance is real and the transformer is inserted at a distance from the load equal to an even multiple of $\lambda/4$, then

$$Z_A = Z_R \quad ; \qquad d_1 = 2n\frac{\lambda}{4} = n\frac{\lambda}{2}$$

but if the distance from the load is an odd multiple of $\lambda/4$

$$Z_A = \frac{Z_{01}^2}{Z_R} \quad ; \qquad d_1 = (2n+1)\frac{\lambda}{4} = n\frac{\lambda}{2} + \frac{\lambda}{4}$$

The input impedance of the impedance transformer after inclusion in the circuit is given by

$$Z_B = \frac{Z_{02}^2}{Z_A}$$

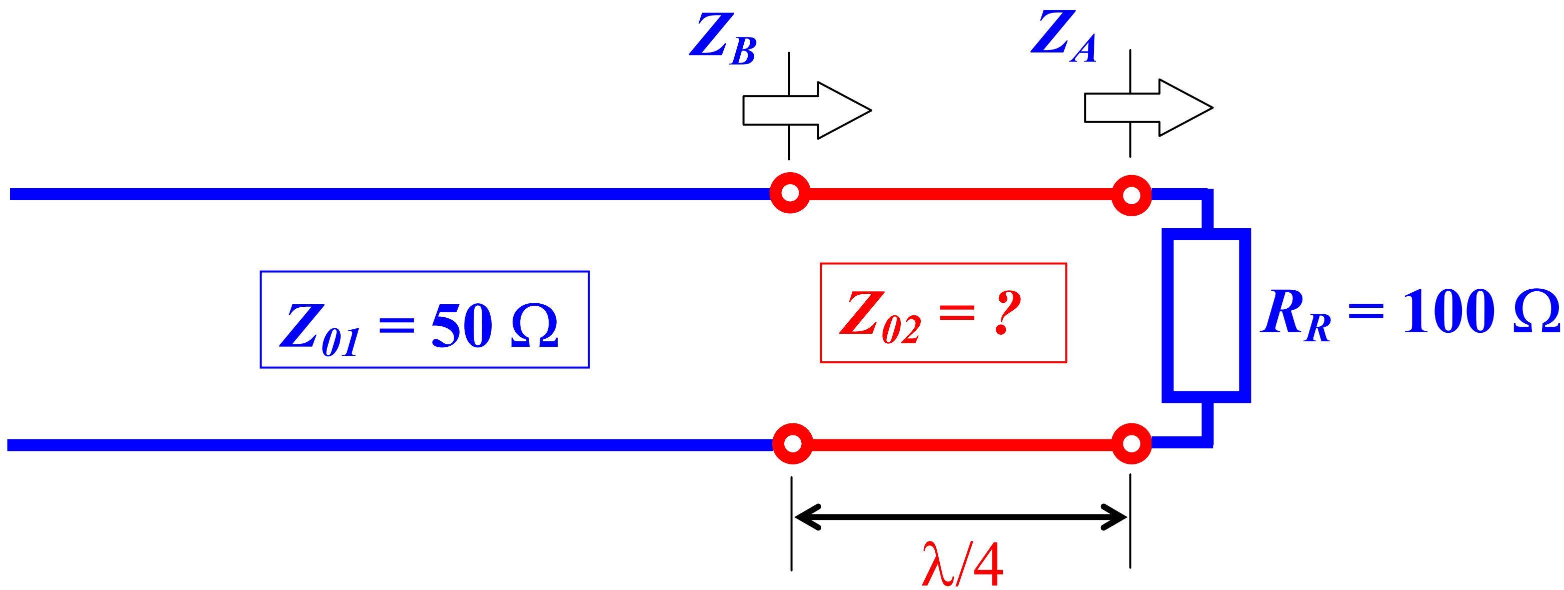
For impedance matching we need

$$Z_{01} = \frac{Z_{02}^2}{Z_A} \quad \Rightarrow \quad Z_{02} = \sqrt{Z_{01} Z_A}$$

The characteristic impedance of the transformer is simply the geometric average between the characteristic impedance of the original line and the load seen by the transformer.

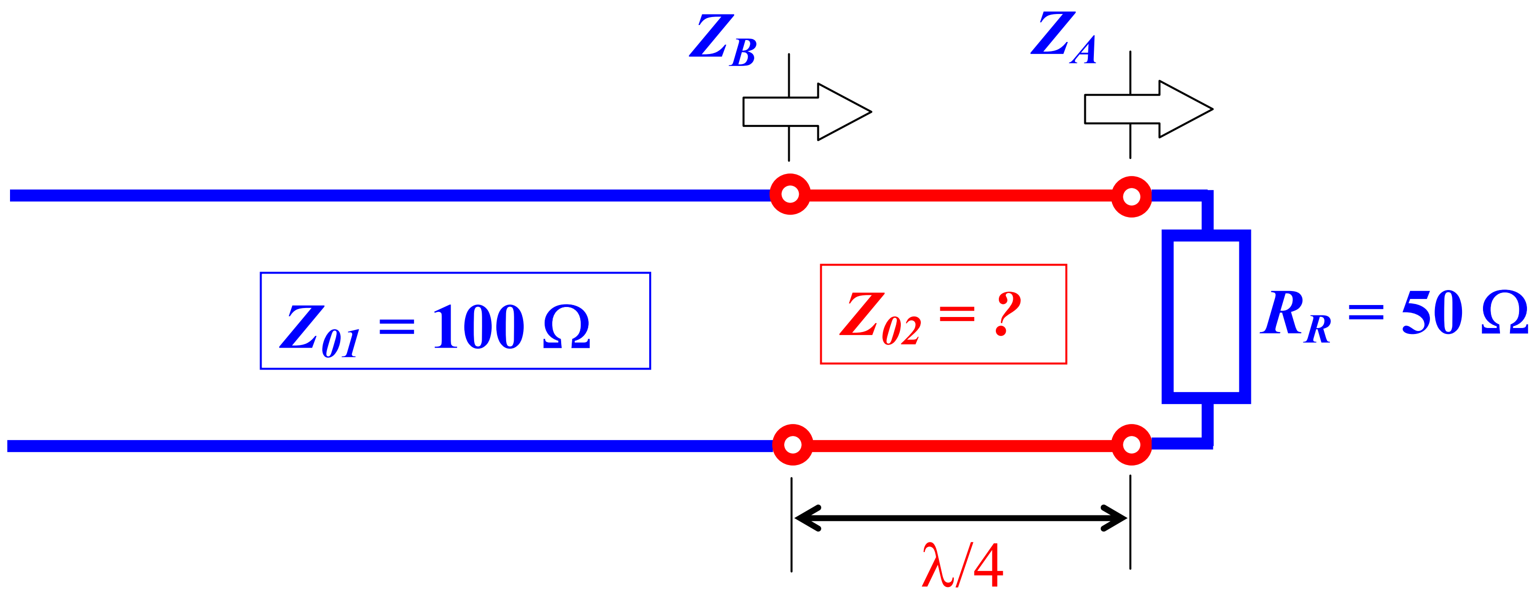
Let's now review some simple examples.

❑ Real Load Impedance



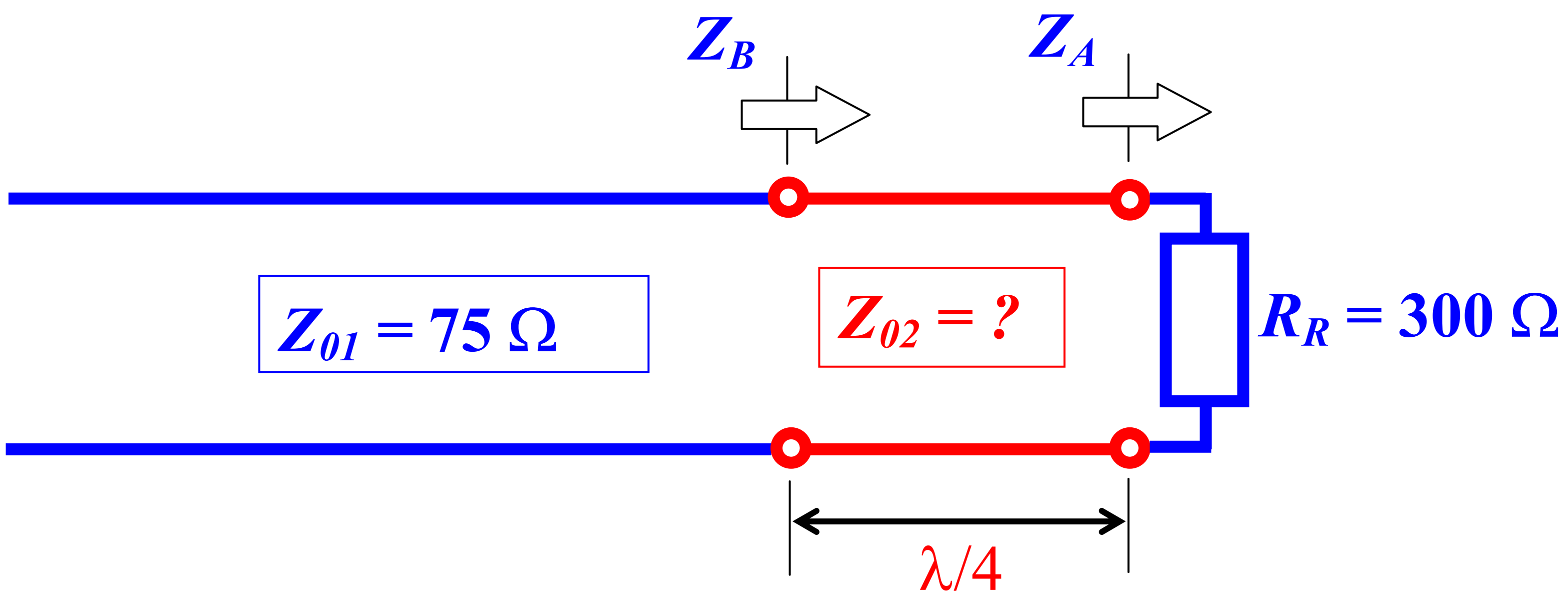
$$Z_B = \frac{Z_{02}^2}{R_R} = Z_{01} \Rightarrow \; Z_{02} = \sqrt{Z_{01} R_R} = \sqrt{50 \cdot 100} \approx 70.71\ \Omega$$

Note that an identical result is obtained by switching Z_{01} and R_R



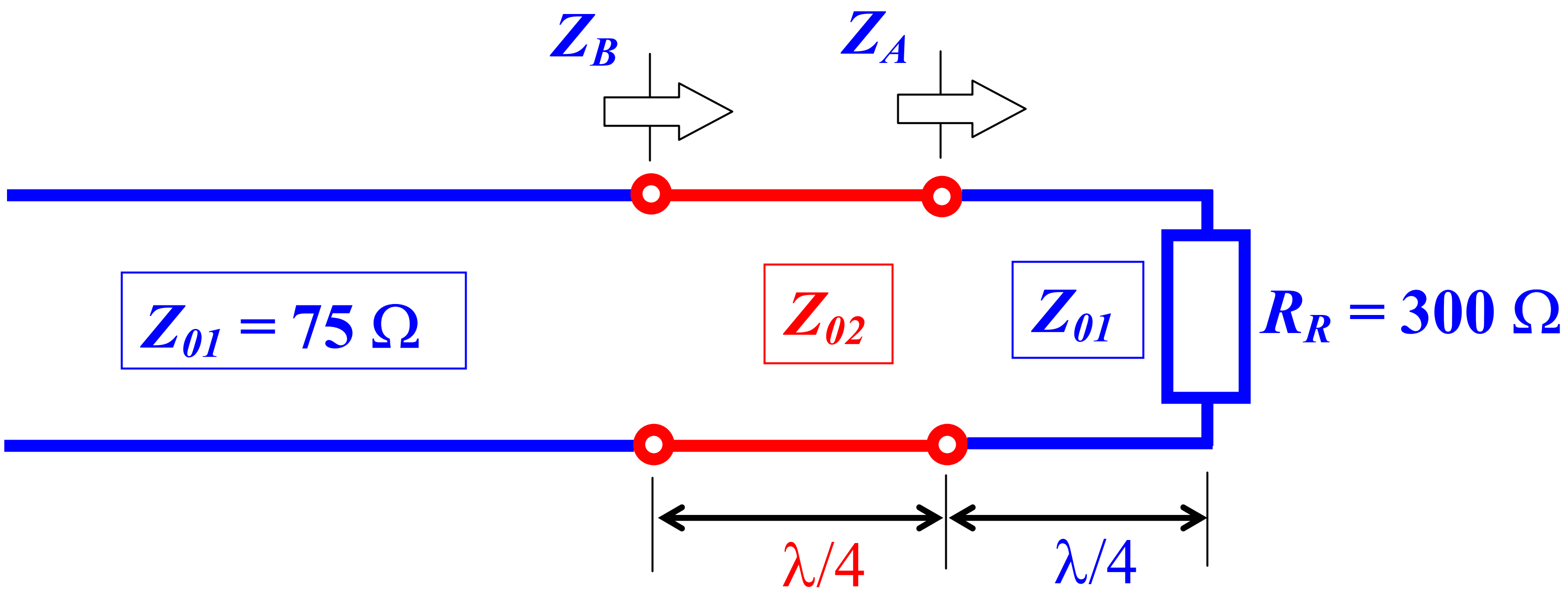
$$Z_B = \frac{Z_{02}^2}{R_R} = Z_{01} \Rightarrow \ Z_{02} = \sqrt{Z_{01} R_R} = \sqrt{100 \cdot 50} \approx 70.71\,\Omega$$

Another real load case



$$Z_B = \frac{Z_{02}^2}{R_R} = Z_{01} \Rightarrow \ Z_{02} = \sqrt{Z_{01} R_R} = \sqrt{75 \cdot 300} = 150\ \Omega$$

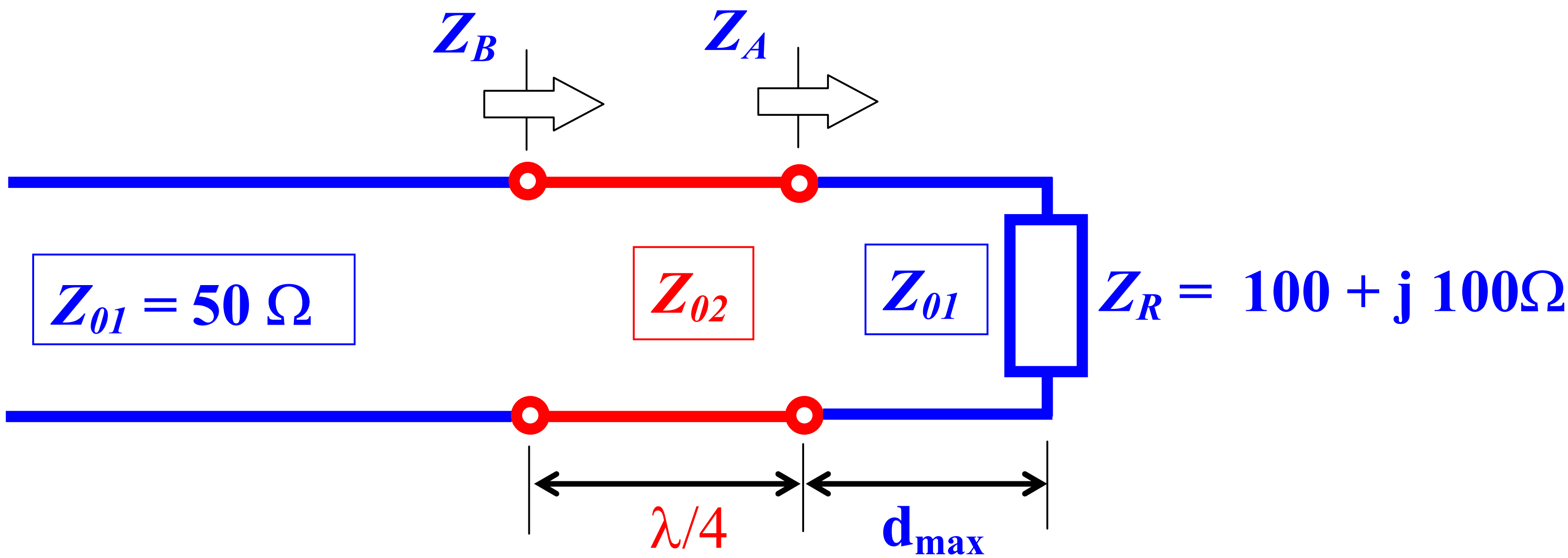
Same impedances as before, but now the transformer is inserted at a distance $\lambda/4$ from the load (voltage minimum in this case)



$$Z_A = \frac{Z_{01}^2}{R_R} = \frac{75^2}{300} = 18.75\,\Omega$$

$$Z_B = \frac{Z_{02}^2}{Z_A} = Z_{01} \Rightarrow \; Z_{02} = \sqrt{Z_{01} Z_A} = \sqrt{75 \cdot 18.75} = 37.5\,\Omega$$

❑ Complex Load Impedance – Transformer at voltage maximum

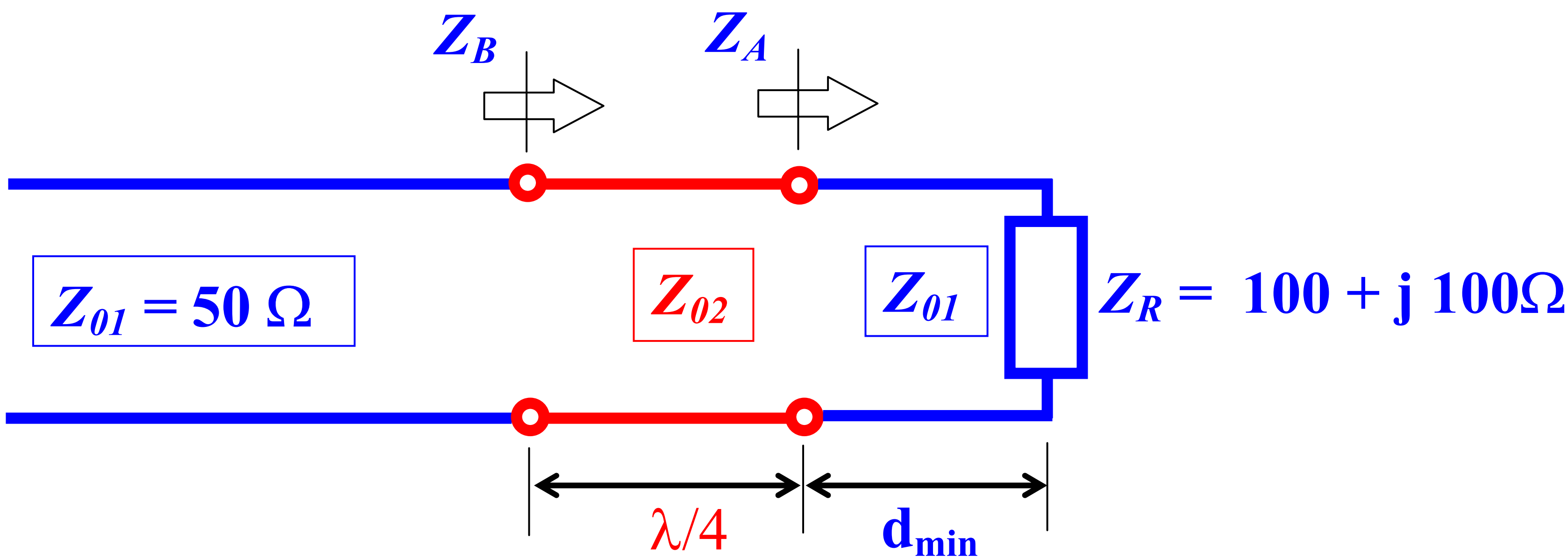


$$|\Gamma_R| = \left|\frac{100 + j100 - 50}{100 + j100 + 50}\right| \approx 0.62$$

$$Z_A = Z_0 \frac{1 + |\Gamma_R|}{1 - |\Gamma_R|} \approx 213.28\Omega$$

$$Z_{02} = \sqrt{Z_{01} Z_A} = \sqrt{50 \cdot 213.28} = 103.27\ \Omega$$

❑ Complex Load Impedance – Transformer at voltage minimum



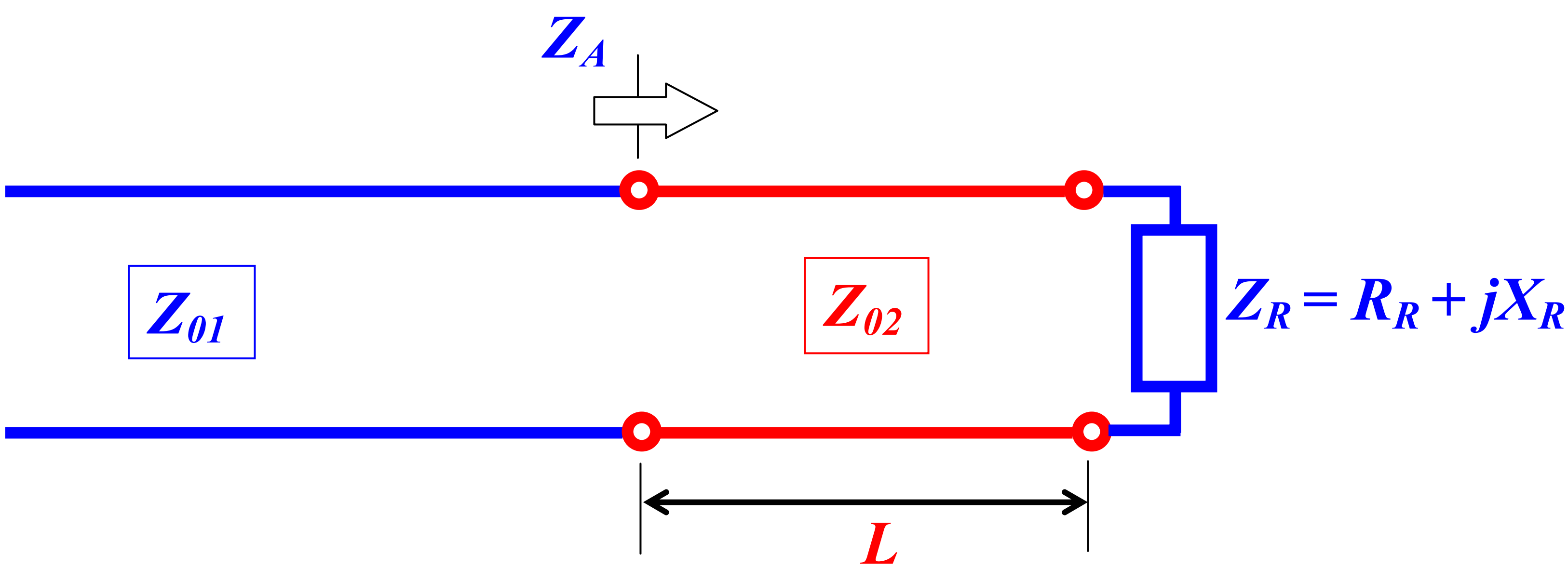
$$\left|\Gamma_R\right| = \left|\frac{100 + j100 - 50}{100 + j100 + 50}\right| \approx 0.62$$

$$Z_A = Z_0 \frac{1 - \left|\Gamma_R\right|}{1 + \left|\Gamma_R\right|} \approx 11.72\ \Omega$$

$$Z_{02} = \sqrt{Z_{01} Z_A} = \sqrt{50 \cdot 11.72} = 24.21\ \Omega$$

- **Generalized Transformer**

If it is not important to realize the impedance transformer with a quarter wavelength line, one may try to select a transmission line with appropriate length and characteristic impedance, such that the input impedance is the required real value



$$Z_{01} = Z_A = Z_{02} \frac{R_R + jX_R + jZ_{02} \tan(\beta L)}{Z_{02} + j\left(R_R + jX_R\right) \tan(\beta L)}$$

After separation of real and imaginary parts we obtain the equations

$$Z_{02}(Z_{01} - R_R) = Z_{01} X_R \tan(\beta \text{ L})$$

$$\tan(\beta \text{ L}) = \frac{Z_{02} X_R}{Z_{01} R_R - Z_{02}^2}$$

with final solution

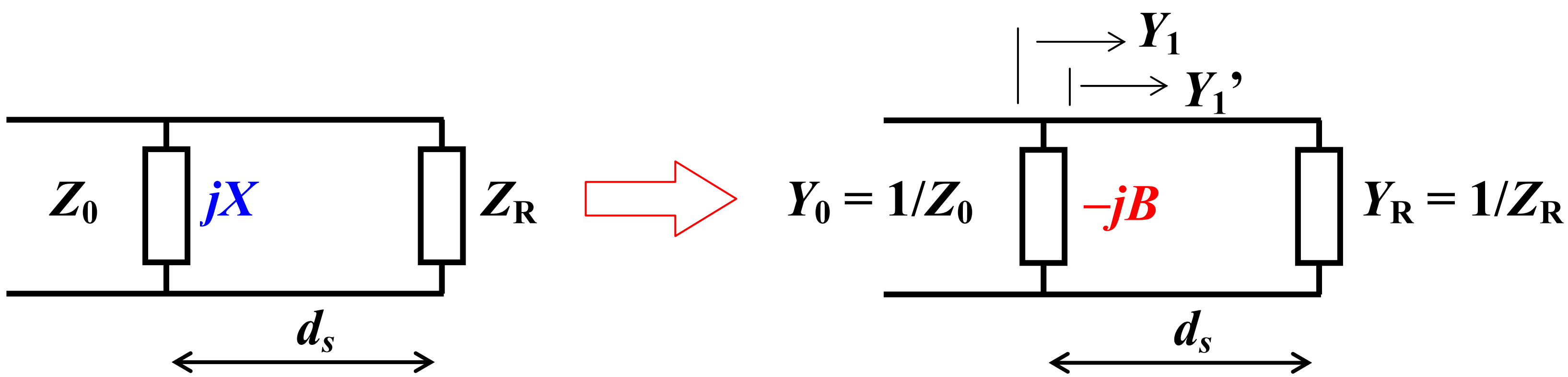
$$Z_{02} = \frac{\sqrt{Z_{01} R_R - R_R^2 - X_R^2}}{\sqrt{1 - R_R / Z_{01}}}$$

$$\tan(\beta \text{ L}) = \frac{\sqrt{(1 - R_R / Z_{01})\left(Z_{01} R_R - R_R^2 - X_R^2 \right)}}{X_R}$$

The transformer can be realized as long as the result for Z_{02} is real. Note that this is also a narrow band approach.

Matching II – Shunt Admittance

We wish to insert a parallel (shunt) reactance on the transmission line to obtain impedance matching. Since the design involves a parallel circuit, it is more convenient to consider admittances:



The shunt may be inserted at locations d_s where the real part of the line admittance is equal to the characteristic admittance Y_0

$$Y_1' = Y_0 + jB$$

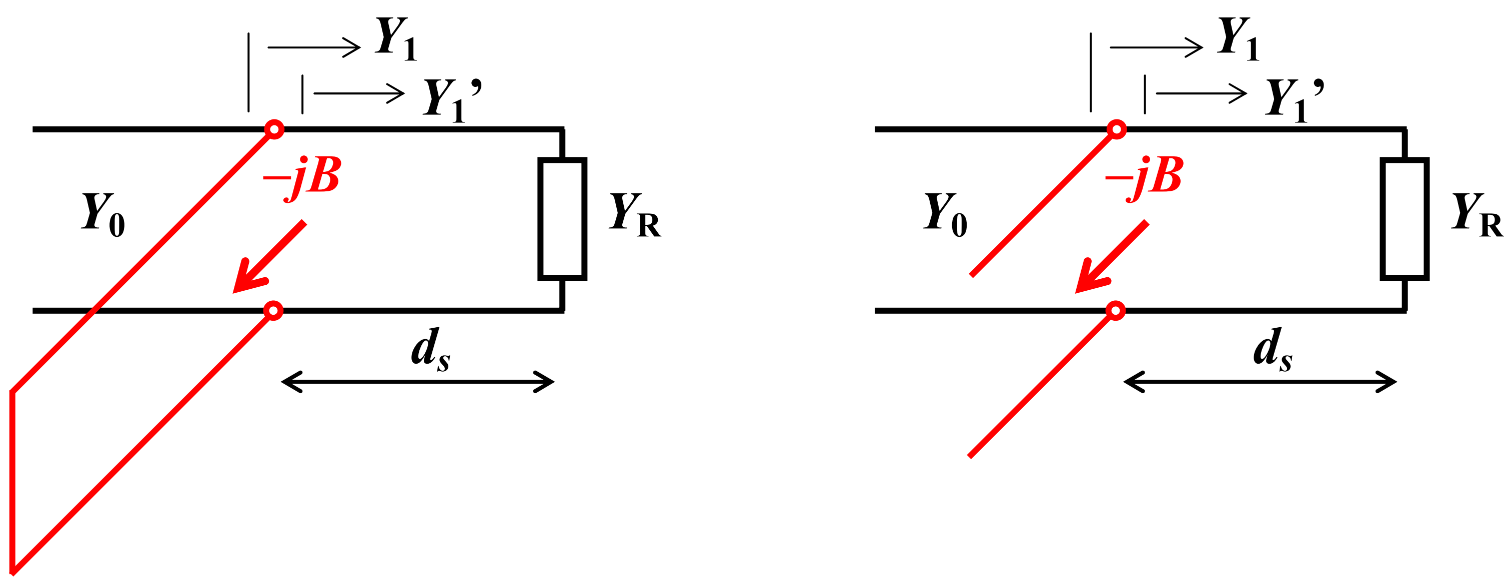
Matching is obtained by using a shunt susceptance $-jB$ so that

$$Y_1 = [Z(d_s)]^{-1} - jB = Y_1' - jB = Y_0$$

To solve this design problem, we need to find the suitable locations d_s (where the real part of the line admittance is equal to Y_0) and the corresponding values of the shunt susceptance $-B$.

The shunt element may be also realized by inserting a segment of transmission line of appropriate length, called a stub.

In order to obtain a pure susceptance, the stub element may consist of a short-circuited or an open-circuited transmission line with input admittance $-jB$.



The line admittance at location d_s can be expressed as a function of reflection coefficient

$$Y_1' = Y_0 + jB = \left(Z_0 \frac{1+\Gamma(d_s)}{1-\Gamma(d_s)} \right)^{-1} = Y_0 \frac{1-\Gamma(d_s)}{1+\Gamma(d_s)}$$

For more general results, we introduce *normalization*:

$$\frac{Y_1'}{Y_0} = y_1' = 1 + jb = \frac{1-\Gamma(d_s)}{1+\Gamma(d_s)} = \text{normalized admittance}$$

$$b = \text{normalized susceptance}$$

Then, the line reflection coefficient can be expressed in terms of b

$$y_1' = \frac{1-\Gamma(d_s)}{1+\Gamma(d_s)} \quad \Rightarrow \quad \Gamma(d_s) = \frac{1-y_1'}{1+y_1'} = \frac{1-(1+jb)}{1+1-jb} = \frac{-jb}{2+jb}$$

Since we know that $\Gamma(d_s) = \Gamma_R \exp(-j2\beta d_s)$

$$\Gamma_R = |\Gamma_R| \exp(j\theta) = \frac{-jb}{2+jb} \exp(j2\beta d_s)$$

$$= \frac{|b|}{\sqrt{4+b^2}} \exp\left(\mp j\pi/2 - \tan^{-1}(b/2)\right) \exp(j2\beta d_s + j2n\pi)$$

$-$ for $b > 0$; $+$ for $b < 0$

Added to account for periodic behavior

The absolute value of the load reflection coefficient provides *b*

$$|\Gamma_R| = \frac{|b|}{\sqrt{4+b^2}} \Rightarrow b^2 = \frac{4|\Gamma_R|^2}{1-|\Gamma_R|^2} \Rightarrow b = \pm \frac{2|\Gamma_R|}{\sqrt{1-|\Gamma_R|^2}}$$

$$B = b \cdot Y_0 = b / Z_0$$

Finally, the phase of the load reflection coefficient yields d_s

$$\angle \Gamma_R = \theta = \mp \pi/2 + 2\beta\, d_s - \tan^{-1}(b/2) + 2\,n\,\pi$$

$$\Rightarrow \quad 2\beta\, d_s = \frac{4\pi}{\lambda} d_s = \theta \mp \pi/2 - \tan^{-1}(b/2) - 2\,n\,\pi$$

$$d_s = \frac{\lambda}{4\pi}(\theta \pm \pi/2 + \tan^{-1}(b/2) - 2\,n\,\pi)$$

⇧

$+$ for $b > 0$; $\quad -$ for $b < 0$

The last term accounting for periodic behavior of the solution gives

$$\frac{\lambda}{4\pi} 2\,n\,\pi = n\frac{\lambda}{2}$$

indicating that the solutions repeat every $\lambda/2$ along the line.