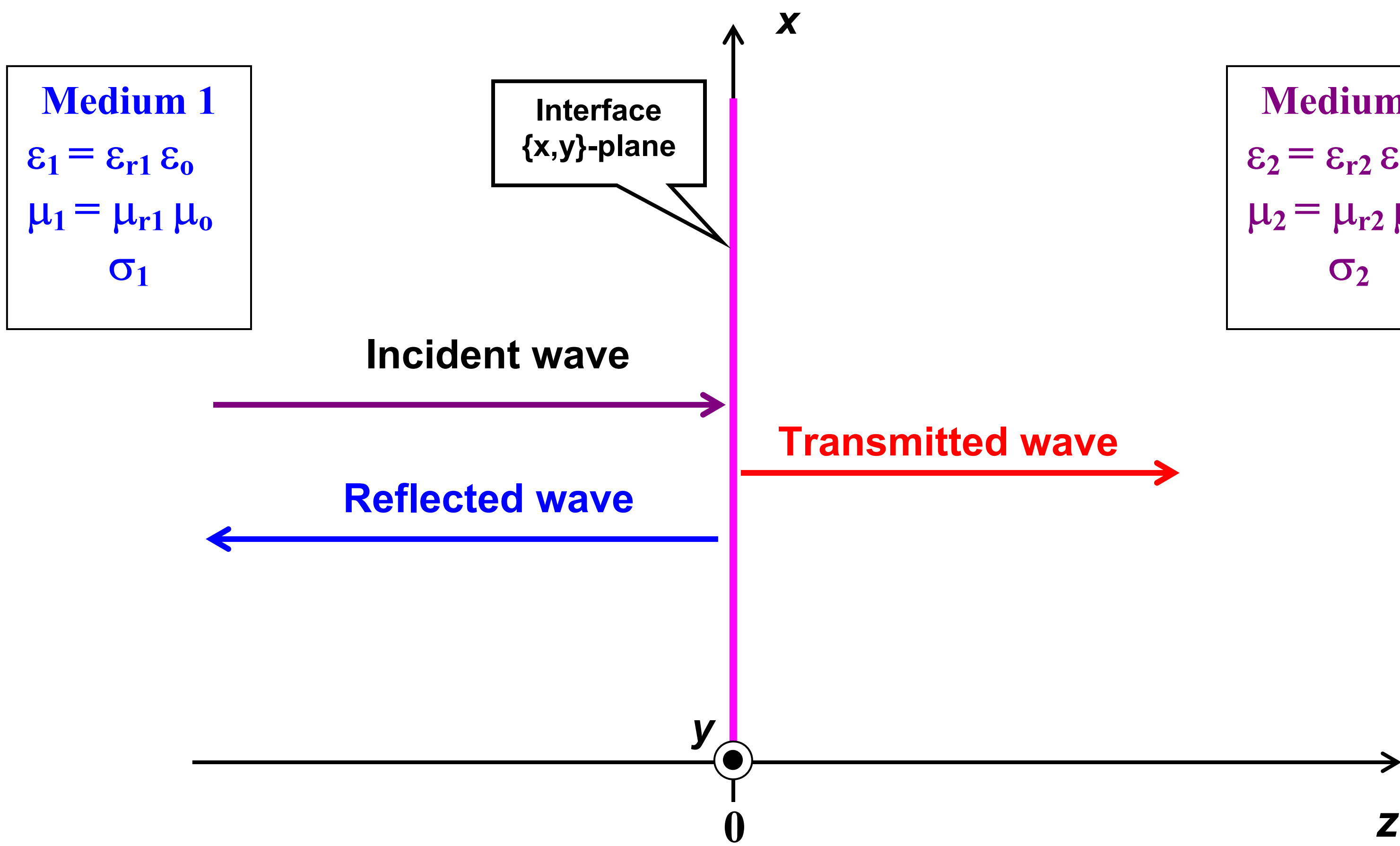
Normal Incidence on an Interface

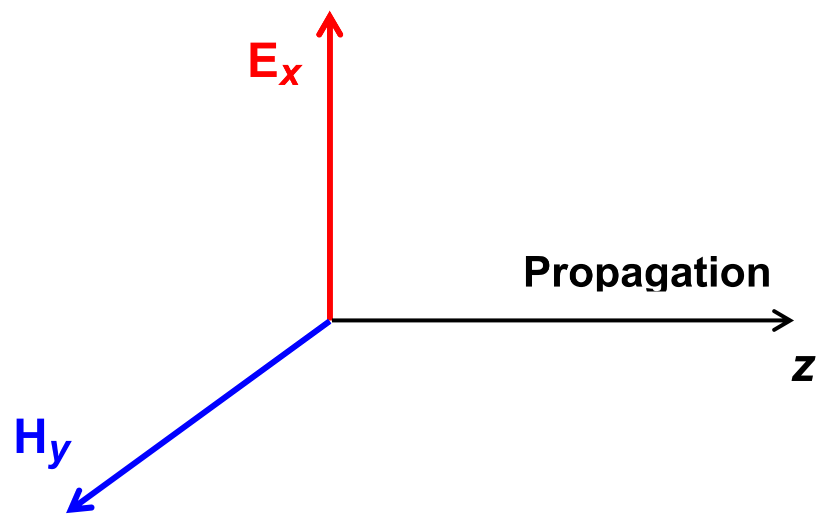
Consider a planar interface between two unbounded media, and a uniform plane wave with normal incidence on the interface.



Because of the medium discontinuity, the incident wave experiences a partial reflection at the interface.

In medium 2, only a forward transmitted wave exists

The total fields at the interface must satisfy the boundary conditions for electromagnetic fields. Without loss of generality, we assume the following orientation for the electromagnetic fields of the waves



Recalling the solution for Helmholtz equation, the phasor fields in the medium 1 can be written as

$$\mathbf{E}_1(z) = \mathbf{E}_1^+ \exp(-\gamma_1 z) + \mathbf{E}_1^- \exp(\gamma_1 z)$$

$$\mathbf{H}_1(z) = \mathbf{H}_1^+ \exp(-\gamma_1 z) + \mathbf{H}_1^- \exp(\gamma_1 z)$$

$$= \frac{1}{\eta_1}\left(\mathbf{E}_1^+ \exp(-\gamma_1 z) - \mathbf{E}_1^- \exp(\gamma_1 z)\right)$$

Total Field | **Incident wave** | **Reflected wave**

$$\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)} \qquad \eta_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_1}}$$

The forward transmitted wave in medium 2 is given by

$$E_{2x}(z) = E_2^+ \exp(-\gamma_2 z)$$

$$H_{2y}(z) = H_2^+ \exp(-\gamma_2 z)$$

$$= \frac{1}{\eta_2} E_2^+ \exp(-\gamma_2 z)$$

Total Field — **Transmitted wave**

$$\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)} \qquad \eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$$

Both fields are parallel to the interface. The boundary conditions indicate that the total fields are continuous at the interface. Note that we are assuming a finite conductivity, therefore no surface current exists and the tangent magnetic field is also continuous.

The interface is located at $z = 0$ so all exponentials are equal to 1:

$$E_{1x}(z=0) = E_{2x}(z=0) \quad \Rightarrow \quad E_1^+ + E_1^- = E_2^+$$

$$H_{1y}(z=0) = H_{2y}(z=0) \quad \Rightarrow \quad \frac{1}{\eta_1}\left(E_1^+ - E_1^-\right) = \frac{1}{\eta_2} E_2^+$$

Assuming that the amplitude E_1^+ of the incident wave is known, we have two unknowns E_1^- and E_2^+. In order to obtain a general result, it is convenient to solve the equations above in terms of reflection coefficient (E_1^-/E_1^+) and transmission coefficient (E_2^+/E_1^+).

Reflection Coefficient

$$\Gamma_E = \frac{E_1^-}{E_1^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

This is similar to the voltage load reflection coefficient found for a transmission line, if one considers the following analogy

Medium 1 ⇔ Transmission Line

η_1 ⇔ Z_0 **Characteristic Impedance**

Medium 2 ⇔ Load

η_2 ⇔ Z_R **Load Impedance**

For the magnetic field

$$\Gamma_H = \frac{H_1^-}{H_1^+} = \frac{-E_1^- / \eta_1}{E_1^+ / \eta_1} = -\frac{E_1^-}{E_1^+} = -\Gamma_E$$

Transmission Coefficient

$$\tau_E = \frac{E_2^+}{E_1^+} = \frac{E_1^+ + E_1^-}{E_1^+} = 1 + \frac{E_1^-}{E_1^+}$$

$$= 1 + \Gamma_E = \frac{2\eta_2}{\eta_2 + \eta_1}$$

For the magnetic field we have

$$\tau_H = \frac{H_2^+}{H_1^+} = \frac{H_1^+ + H_1^-}{H_1^+} = 1 + \frac{H_1^-}{H_1^+} = 1 - \frac{E_1^-}{E_1^+} = 1 - \Gamma_E$$

NOTE: The reflection and transmission coefficients for the fields are in general complex quantities.

Special cases

Matched Impedances

$$\eta_1 = \eta_2 \quad \Rightarrow \quad \Gamma_E = 0 \quad \text{and} \quad \tau_E = 1$$

In this case we have total transmission into medium 2 and no reflection.

Medium 2 = Perfect Conductor

$$\sigma_2 \rightarrow \infty \quad \Rightarrow \quad \eta_2 = 0 \quad \Rightarrow \quad \Gamma_E = -1 \quad \text{and} \quad \tau_E = 0$$

The wave experiences total reflection, consistent with the fact that the fields must be zero inside the perfect conducting medium. This case is analogous to a line with a short circuit load. The total electric fields at the interface is

$$\mathbf{E}_1^+ + \mathbf{E}_1^- = \mathbf{E}_1^+ + \Gamma_E \mathbf{E}_1^+ = \mathbf{E}_1^+ - \mathbf{E}_1^+ = \mathbf{E}_2^+ = 0$$

Perfect Dielectric Media

$$\sigma_1 = 0 \quad \Rightarrow \quad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \text{Real}$$

$$\sigma_2 = 0 \quad \Rightarrow \quad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \text{Real}$$

Usually,

$$\mu_1 = \mu_2 = \mu_o$$

$$\Rightarrow \quad \Gamma_E = \frac{\sqrt{\mu_o / \varepsilon_2} - \sqrt{\mu_o / \varepsilon_1}}{\sqrt{\mu_o / \varepsilon_2} + \sqrt{\mu_o / \varepsilon_1}} = \frac{1 - \sqrt{\varepsilon_2 / \varepsilon_1}}{1 + \sqrt{\varepsilon_2 / \varepsilon_1}} = \text{Real}$$

$$\tau_E = 1 + \Gamma = 1 + \frac{1 - \sqrt{\varepsilon_2 / \varepsilon_1}}{1 + \sqrt{\varepsilon_2 / \varepsilon_1}} = \frac{2}{1 + \sqrt{\varepsilon_2 / \varepsilon_1}} = \text{Real}$$

Power flow

Assuming dielectric media (no-loss) for simplicity, the time-average power associated with the incident wave and the reflected wave is

$$\left\langle \vec{P}(t) \right\rangle_{in} = \frac{1}{2}\mathrm{Re}\left\{ \vec{\mathbf{E}} \times \vec{\mathbf{H}}^* \right\} = \frac{1}{2}\frac{\left|\mathbf{E}_1^+\right|^2}{\eta_1}$$

$$\left\langle \vec{P}(t) \right\rangle_{refl} = \frac{1}{2}\frac{\left|\mathbf{E}_1^+\right|^2}{\eta_1}\left|\Gamma_{\mathbf{E}}\right|^2$$

The power reflection coefficient is

$$\Gamma_{\mathbf{P}} = \frac{\left\langle \vec{P}(t) \right\rangle_{refl}}{\left\langle \vec{P}(t) \right\rangle_{in}} = \left|\Gamma_{\mathbf{E}}\right|^2$$

The time-average power flow transmitted in medium 2 is

$$\left\langle \vec{P}(t) \right\rangle_{trans} = \frac{1}{2} \frac{\left| E_2^+ \right|^2}{\eta_2}$$

Also

$$\left\langle \vec{P}(t) \right\rangle_{trans} = \left\langle \vec{P}(t) \right\rangle_{in} - \left\langle \vec{P}(t) \right\rangle_{refl} = \frac{1}{2} \frac{\left| E_1^+ \right|^2}{\eta_1} \left(1 - \left| \Gamma_E \right|^2 \right)$$

since power flow normal to the interface must be continuous

$$\Rightarrow \qquad \frac{1}{2} \frac{\left| E_2^+ \right|^2}{\eta_2} = \frac{1}{2} \frac{\left| E_1^+ \right|^2}{\eta_1} \left(1 - \left| \Gamma_E \right|^2 \right)$$

The power transmission coefficient is

$$\tau_{\mathbf{P}} = \frac{\left\langle \vec{P}(t) \right\rangle_{trans}}{\left\langle \vec{P}(t) \right\rangle_{in}} = \left| \tau_{\mathbf{E}} \right|^{2} \frac{\eta_1}{\eta_2}$$

$$= \frac{\left\langle \vec{P}(t) \right\rangle_{in} - \left\langle \vec{P}(t) \right\rangle_{refl}}{\left\langle \vec{P}(t) \right\rangle_{in}} = 1 - \left| \Gamma_{\mathbf{E}} \right|^{2}$$

Note that, as a consequence of power conservation, from the results above one gets

$$\Gamma_{\mathbf{P}} + \tau_{\mathbf{P}} = \left| \Gamma_{\mathbf{E}} \right|^{2} + 1 - \left| \Gamma_{\mathbf{E}} \right|^{2} = 1$$

NOTE: the reflection and transmission coefficients for the time-average power flow are always real.

