Power Flow in Electromagnetic Waves

The time-dependent power flow density of an electromagnetic wave is given by the instantaneous Poynting vector

$$\vec{P}(t) = \vec{E}(t) \times \vec{H}(t)$$

For time-varying fields it is important to consider the time-average power flow density

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{T}\int_0^T \vec{P}(t)\, dt = \frac{1}{T}\int_0^T \vec{E}(t) \times \vec{H}(t)\, dt$$

where T is the period of observation.

Consider time-harmonic fields represented in terms of their phasors

$$\vec{E}(t) = \mathrm{Re}\left\{ \vec{\mathrm{E}} \exp(j\omega t) \right\} = \mathrm{Re}\{\vec{\mathrm{E}}\} \cos\omega t - \mathrm{Im}\{\vec{\mathrm{E}}\} \sin\omega t$$

$$\vec{H}(t) = \mathrm{Re}\left\{ \vec{\mathrm{H}} \exp(j\omega t) \right\} = \mathrm{Re}\{\vec{\mathrm{H}}\} \cos\omega t - \mathrm{Im}\{\vec{\mathrm{H}}\} \sin\omega t$$

The time-dependent Poynting vector can be expressed as the sum of the cross-products of the components

$$\begin{aligned} \vec{E}(t) \times \vec{H}(t) &= \mathrm{Re}\{\vec{\mathrm{E}}\} \times \mathrm{Re}\{\vec{\mathrm{H}}\} \cos^2 \omega t \\ &+ \mathrm{Im}\{\vec{\mathrm{E}}\} \times \mathrm{Im}\{\vec{\mathrm{H}}\} \sin^2 \omega t \\ &- \left(\mathrm{Re}\{\vec{\mathrm{E}}\} \times \mathrm{Im}\{\vec{\mathrm{H}}\} + \mathrm{Im}\{\vec{\mathrm{E}}\} \times \mathrm{Re}\{\vec{\mathrm{H}}\} \right) \cos\omega t \sin\omega t \end{aligned}$$

(Note that: $\cos\omega t \sin\omega t = \frac{1}{2} \sin 2\omega t$ **)**

The time-average power flow density can be obtained by integrating the previous result over a period of oscillation T. The pre-factors containing field phasors do not depend on time, therefore we have to solve for the following integrals:

$$\frac{1}{T}\int_0^T \cos^2\omega t\, dt = \frac{1}{T}\left[\frac{t}{2} + \frac{\sin 2\omega t}{4\omega}\right]_0^T = \frac{1}{2}$$

$$\frac{1}{T}\int_0^T \sin^2\omega t\, dt = \frac{1}{T}\left[\frac{t}{2} - \frac{\sin 2\omega t}{4\omega}\right]_0^T = \frac{1}{2}$$

$$\frac{1}{T}\int_0^T \cos\omega t \cdot \sin\omega t\, dt = \frac{1}{T}\left[\frac{\sin^2\omega t}{2\omega}\right]_0^T = 0$$

The final result for the time-average power flow density is given by

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{T}\int_0^T \vec{E}(t)\times\vec{H}(t)\, dt$$

$$= \frac{1}{2}\left(\mathrm{Re}\{\vec{\mathrm{E}}\}\times\mathrm{Re}\{\vec{\mathrm{H}}\} + \mathrm{Im}\{\vec{\mathrm{E}}\}\times\mathrm{Im}\{\vec{\mathrm{H}}\}\right)$$

Now, consider the following cross product of phasor vectors

$$\vec{\mathrm{E}}\times\vec{\mathrm{H}}^{*} = \mathrm{Re}\{\vec{\mathrm{E}}\}\times\mathrm{Re}\{\vec{\mathrm{H}}\} + \mathrm{Im}\{\vec{\mathrm{E}}\}\times\mathrm{Im}\{\vec{\mathrm{H}}\}$$

$$+\, j\left(\mathrm{Im}\{\vec{\mathrm{E}}\}\times\mathrm{Re}\{\vec{\mathrm{H}}\} - \mathrm{Re}\{\vec{\mathrm{E}}\}\times\mathrm{Im}\{\vec{\mathrm{H}}\}\right)$$

By combining the previous results, one can obtain the following time average rule

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{T}\int_0^T \vec{E}(t)\times\vec{H}(t)\,dt = \frac{1}{2}\mathrm{Re}\left\{\vec{\mathrm{E}}\times\vec{\mathrm{H}}^*\right\}$$

We also call complex Poynting vector the quantity

$$\vec{\mathrm{P}} = \frac{1}{2}\vec{\mathrm{E}}\times\vec{\mathrm{H}}^*$$

NOTE: the complex Poynting vector is not the phasor of the time-dependent power nor that of the time-average power density!

$$\left\langle \vec{P}(t) \right\rangle = \mathrm{Re}\left\{\vec{\mathrm{P}}\right\}$$

(don't try ~~$\vec{P}(t) = \mathrm{Re}\left\{\vec{\mathrm{P}}\exp(j\omega t)\right\}$~~ **)**

Phasor notation cannot be applied to the product of two time-harmonic functions (e.g., $P(t)$), even if they have same frequency.

Consider a 1-D electro-magnetic wave moving along the z-direction, with a specified electric field amplitude E_o

$$\mathrm{E}_x(z) = E_o \exp(-\alpha z)\exp(-j\beta z)$$

$$\mathrm{H}_y(z) = \frac{E_o}{|\eta|}\exp(-\alpha z)\exp(-j\beta z)\exp(-j\tau)$$

The time-average power flow density is

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left\{\vec{\mathrm{E}} \times \vec{\mathrm{H}}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{E_o e^{-\alpha z} e^{-j\beta z} \frac{E_o^*}{|\eta|} e^{-\alpha z} e^{j\beta z} e^{j\tau}\right\}$$

$$= \frac{1}{2}|E_o|^2 \frac{e^{-2\alpha z}}{|\eta|}\mathrm{Re}\left\{e^{j\tau}\right\} = \frac{1}{2}|E_o|^2 \frac{e^{-2\alpha z}}{|\eta|}\cos\tau$$

Power in a lossy medium decays as $\exp(-2\alpha z)$!

Consider the same wave, with a specified amplitude for the magnetic field

$$\mathrm{H}_y(z) = H_o \exp(-\alpha z)\exp(-j\beta z)$$

$$\mathrm{E}_x(z) = |\eta| H_o \exp(-\alpha z)\exp(-j\beta z)\exp(j\tau)$$

The time-average power flow density is expressed as

$$\left\langle \vec{P}(t) \right\rangle = \frac{1}{2}\mathrm{Re}\left\{ |\eta| H_o e^{-\alpha z} e^{-j\beta z} H_o^* e^{-\alpha z} e^{j\beta z} e^{j\tau} \right\}$$

$$= \frac{1}{2}|\eta| |H_o|^2 e^{-2\alpha z} \cos\tau$$

If α is the attenuation constant for the electromagnetic fields

$\Rightarrow 2\alpha$ is the attenuation constant for power flow.

If the wave is generated by an infinitesimally thin sheet of uniform current J_{so} (embedded in an infinite material with conductivity σ) we have for propagation along the positive z-direction (normal to the plane of the current sheet):

$$\left| H_o \right| = \frac{J_{so}}{2} \qquad \left| E_o \right| = \left| \eta \right| \frac{J_{so}}{2}$$

$$\left\langle \vec{P}(t) \right\rangle = \frac{J_{so}^2}{8} \left| \eta \right| e^{-2\alpha z} \cos \tau$$

For this ideal case, an identical wave exists, propagating along the negative z-direction and carrying the same amount of power.

Poynting Theorem

Consider the divergence of the time-dependent power flow density

$$\nabla \cdot \vec{P}(t) = \nabla \cdot \left(\vec{E}(t) \times \vec{H}(t) \right) = \vec{H}(t) \cdot \nabla \times \vec{E}(t) - \vec{E}(t) \cdot \nabla \times \vec{H}(t)$$

The curls can be expressed by using Maxwell's equations

$$\nabla \cdot \vec{P}(t) = -\mu\, \vec{H}(t) \cdot \frac{\partial \vec{H}}{\partial t} - \sigma \vec{E}(t) \cdot \vec{E}(t) - \varepsilon \vec{E}(t) \cdot \frac{\partial \vec{E}}{\partial t}$$

$$= -\, \sigma E^2(t) \quad - \frac{\partial}{\partial t}\left(\frac{1}{2} \varepsilon\, E^2(t) \right) \quad - \frac{\partial}{\partial t}\left(\frac{1}{2} \mu\, H^2(t) \right)$$

Density of dissipated power	Rate of change of stored electric energy density	Rate of change of stored magnetic energy density

This is the differential form of Poynting Theorem.

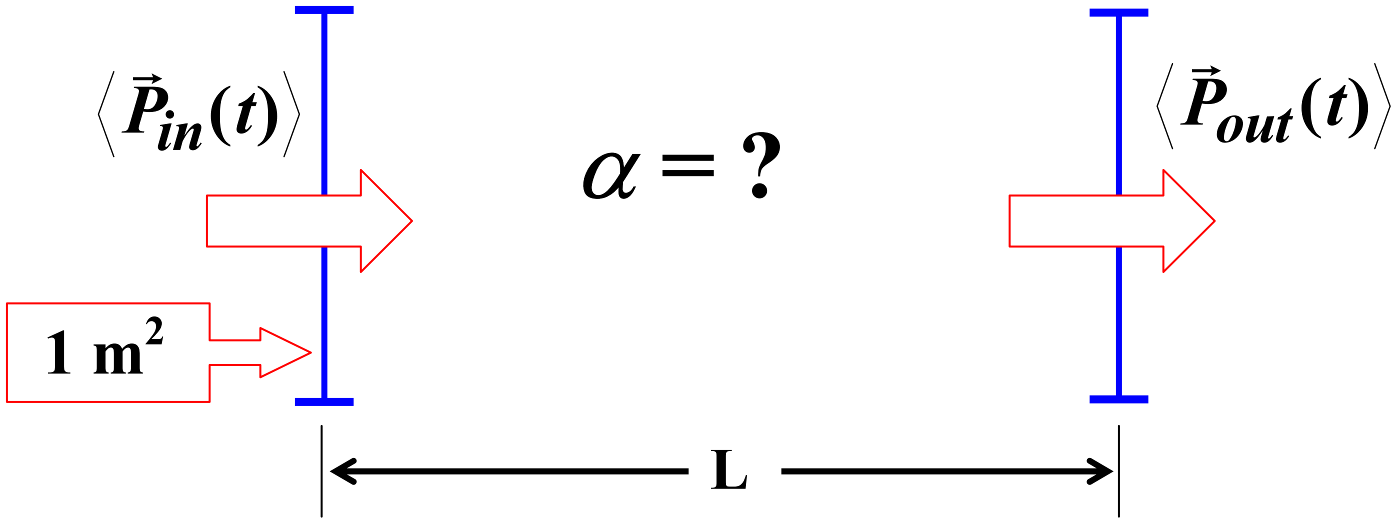
Now, integrate the divergence of the time-dependent power over a specified volume V to obtain the integral form of Poynting theorem

$$\int_V \nabla \cdot \vec{P}(t)\, dV = \oiint_S \vec{P}(t) \cdot ds = \text{Power Flux through S}$$

$$= -\int_V \sigma E^2(t)\, dV - \frac{\partial}{\partial t}\int_V \frac{1}{2}\varepsilon\, E^2(t)\, dV - \frac{\partial}{\partial t}\int_V \frac{1}{2}\mu\, H^2(t)\, dV$$

Power dissipated in volume	Rate of change of electric energy stored in volume	Rate of change of magnetic energy stored in volume

Typical applications



$$\langle \vec{P}_{out}(t) \rangle = \langle \vec{P}_{in}(t) \rangle \exp(-2\alpha L) \quad \left[\frac{\text{Watts}}{\text{m}^2} \right]$$

$$\Rightarrow \alpha = -\frac{1}{2L} \ln\left(\frac{\langle \vec{P}_{out}(t) \rangle}{\langle \vec{P}_{in}(t) \rangle} \right) \quad \left[\frac{\text{Nepers}}{\text{m}} \right]$$

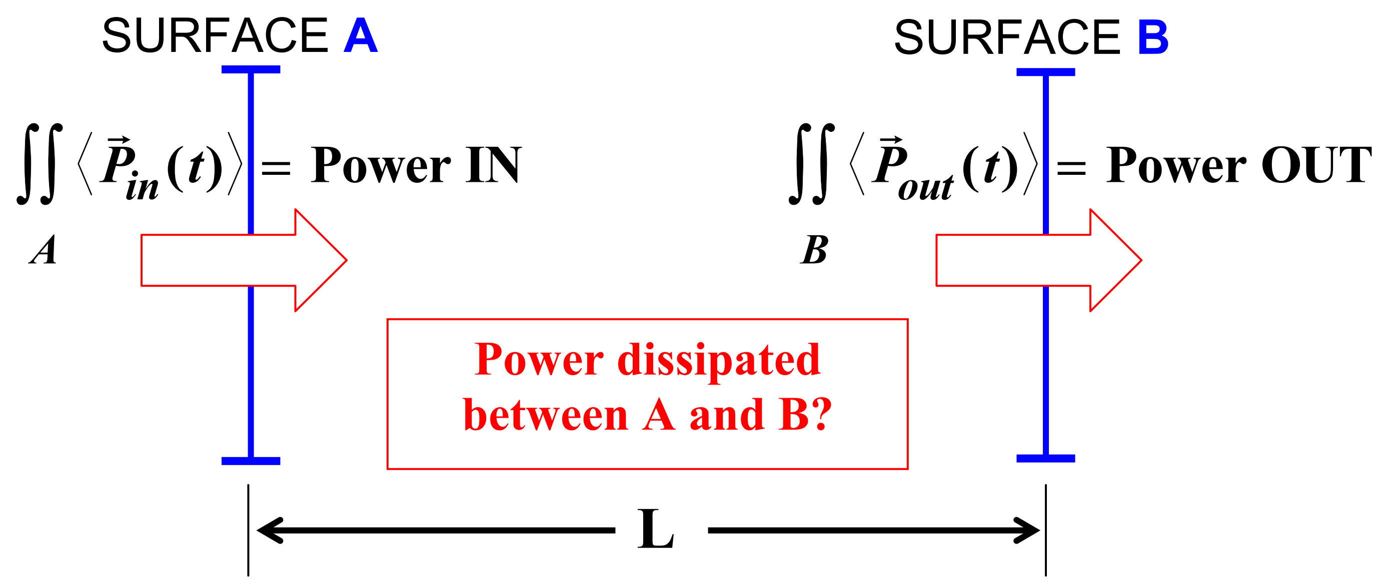
Example:

$$\left\langle \vec{P}_{in}(t) \right\rangle = 30 \left[\frac{\text{Watts}}{\text{m}^2} \right]; \; \left\langle \vec{P}_{out}(t) \right\rangle = 5 \left[\frac{\text{Watts}}{\text{m}^2} \right]; \; L = 20 \text{ m}$$

$$\Rightarrow \quad \alpha = 0.0448 \left[\frac{\text{Nepers}}{\text{m}} \right]$$

Pay attention to the logarithms:

$$\ln\left(\frac{\left\langle \vec{P}_{out}(t) \right\rangle}{\left\langle \vec{P}_{in}(t) \right\rangle} \right) = -\ln\left(\frac{\left\langle \vec{P}_{in}(t) \right\rangle}{\left\langle \vec{P}_{out}(t) \right\rangle} \right)$$



$$\text{Area} = \text{Area(A)} = \text{Area(B)}$$

$$\langle \text{Power IN} \rangle = \iint_A \langle \vec{P}(t) \rangle_A \, dS = \langle \vec{P}(t) \rangle_A \cdot \text{Area}$$

$$\langle \text{Power OUT} \rangle = \iint_B \langle \vec{P}(t) \rangle_B \, dS = \langle \vec{P}(t) \rangle_B \cdot \text{Area}$$

$$\langle \vec{P}(t) \rangle_B = \langle \vec{P}(t) \rangle_A \exp(-2\alpha L)$$

$$\text{Power dissipated} = \langle \text{Power IN} \rangle - \langle \text{Power OUT} \rangle$$

Example

$$\text{Area} = 5\ \text{m}^2; \quad L = 1.0\ \text{cm}; \quad f = 1.0\ \text{GHz}; \quad E_o = 10\ \text{V/m}$$

$$\varepsilon = \varepsilon_o; \quad \mu = \mu_o; \quad \sigma = 0.45755\ \text{S/m}$$

$$\Rightarrow \frac{\sigma}{\omega\varepsilon} = 8.2244637 \quad \text{General Lossy medium}$$

$$\eta = 130.88\angle 0.725\text{rad} = 130.88\angle 41.534^\circ$$

$$\alpha = 40.0\ \ \text{Ne/m}; \qquad \langle \vec{P}_{in}(t) \rangle = 0.286\ \ \text{W/m}^2;$$

$$\langle \vec{P}_{out}(t) \rangle_B = \langle \vec{P}_{in}(t) \rangle_A \exp(-2\alpha L) = 0.12845\ \ \text{W/m}^2;$$

$$\langle \text{Power IN} \rangle = \text{Area} \cdot \langle \vec{P}_{in}(t) \rangle = 1.43\ \text{W}$$

$$\langle \text{Power OUT} \rangle = \text{Area} \cdot \langle \vec{P}(t) \rangle_B = 0.6423\ \text{W}$$

Power dissipated $= \langle \text{Power IN} \rangle - \langle \text{Power OUT} \rangle = 0.7876\ \ \text{W}$