Electromagnetic Waves in Material Media

In a material medium free charges may be present, which generate a current under the influence of the wave electric field. The current $\mathbf{J_c}$ is related to the electric field $\mathbf{E}$ through the conductivity σ as

$$\vec{\mathbf{J}}_c = \sigma\, \vec{\mathbf{E}}$$

The material may also have specific relative values of dielectric permittivity and magnetic permeability

$$\varepsilon = \varepsilon_r\, \varepsilon_o \qquad\qquad \mu = \mu_r\, \mu_o$$

Maxwell's equations become

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}}$$

$$\nabla \times \vec{\mathbf{H}} = \sigma\,\vec{\mathbf{E}} + j\omega\varepsilon\,\vec{\mathbf{E}} = j\omega(\varepsilon - j\frac{\sigma}{\omega})\vec{\mathbf{E}}$$

In phasor notation, it is as if the material conductivity introduces an imaginary part for the dielectric constant ε. The wave equation for the phasor electric field is given by

$$\nabla \times \nabla \times \vec{\mathbf{E}} = \nabla\nabla \cdot \vec{\mathbf{E}} - \nabla^2 \vec{\mathbf{E}} = -j\omega\mu\,\nabla \times \vec{\mathbf{H}}$$

$$= -j\omega\mu(\vec{\mathbf{J}}_{\mathbf{c}} + j\omega\varepsilon\,\vec{\mathbf{E}})$$

$$\Rightarrow \quad \nabla^2 \vec{\mathbf{E}} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{\mathbf{E}}$$

We have assumed that the net charge density is zero, even if a conductivity is present, so that the electric field divergence is zero.

In 1-D the wave equation is simply

$$\frac{\partial^2 \vec{\mathbf{E}}_x}{\partial z^2} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{\mathbf{E}}_x = \gamma^2 \vec{\mathbf{E}}_x$$

with general solution

$$\mathbf{E}_x(z) = \bar{A}\exp(-\gamma z) + \bar{B}\exp(\gamma z)$$

$$H_y(z) = -\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} = \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}\left(\bar{A}\exp(-\gamma z) - \bar{B}\exp(\gamma z)\right)$$

$$= \frac{1}{\bar{\eta}}\left(\bar{A}\exp(-\gamma z) - \bar{B}\exp(\gamma z)\right)$$

These resemble the voltage and current solutions in lossy transmission lines.

The intrinsic impedance of the medium is defined as

$$\bar{\eta} = |\bar{\eta}|\, e^{j\tau} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

For the propagation constant, one can obtain the real and imaginary parts as

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \alpha + j\beta$$

$$\alpha = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2}$$

$$\beta = \frac{\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2}$$

Phase velocity and wavelength are now functions of frequency

$$v_p = \frac{\omega}{\beta} = \frac{\sqrt{2}}{\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{-1/2}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{\sqrt{2}}{f\sqrt{\mu\varepsilon}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]^{-1/2}$$

The intrinsic impedance of the medium is complex as long as the conductivity is not zero. The phase angle of the intrinsic impedance indicates that electric field and magnetic field are out of phase. Considering only the forward wave solutions

$$\mathrm{E}_x(z) = \bar{A}\exp(-\gamma z) = \bar{A}\exp(-\alpha z)\exp(-j\beta z)$$

$$\mathrm{H}_y(z) = \frac{1}{|\eta|}\bar{A}\exp(-\gamma z - j\tau) = \frac{1}{|\eta|}\bar{A}\exp(-\alpha z)\exp(-j\beta z - j\tau)$$

In time-dependent form

$$E_x(z,t) = \mathrm{Re}\left\{ \left|\bar{A}\right| \exp(j\theta)\exp(-\alpha\, z)\exp(-j\beta\, z)\exp(j\omega\, t)\right\}$$

$$= \left|\bar{A}\right| \exp(-\alpha\, z)\cos(\omega\, t - \beta\, z + \theta)$$

$$H_y(z,t) = \frac{1}{|\eta|}\mathrm{Re}\left\{ \left|\bar{A}\right| \exp(j\theta)\exp(-\alpha\, z)\exp(-j\beta\, z - j\tau)\exp(j\omega\, t)\right\}$$

$$= \frac{1}{|\eta|}\left|\bar{A}\right| \exp(-\alpha\, z)\cos(\omega\, t - \beta\, z + \theta - \tau)$$

where the integration constant has been assumed to be in general a complex quantity as

$$\bar{A} = \left|\bar{A}\right| \exp(j\theta)$$

Classification of materials

Perfect dielectrics - For these materials $\sigma = 0$

Propagation constant

$$\beta = \omega\sqrt{\varepsilon_r\varepsilon_o\,\mu_r\mu_o}$$

$$\alpha = 0$$

Phase velocity

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_r\mu_o\varepsilon_r\varepsilon_o}}$$

Medium Impedance

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu_r\mu_o}{\varepsilon_r\varepsilon_o}}$$

Wavelength

$$\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f} = \frac{1}{f\sqrt{\mu_r\mu_o\varepsilon_r\varepsilon_o}}$$

Imperfect dielectrics – For these materials $\sigma \neq 0$ but $(\sigma/\omega\varepsilon) \ll 1$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

$$\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon} + \ldots$$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \qquad \beta \approx \omega\sqrt{\mu\varepsilon}$$

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon}} \qquad \lambda = \frac{2\pi}{\beta} \approx \frac{1}{f\sqrt{\mu\varepsilon}}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{-1/2} \approx \sqrt{\frac{\mu}{\varepsilon}}$$

If $(\sigma/\omega\varepsilon) << 1$, the errors made in the approximations for α, β, v_p and λ are very small, since only terms of order $(\sigma/\omega\varepsilon)^2$ or higher appear in the expansions. The error is slightly higher fo the medium impedance η since the expansion contains a term of order $(\sigma/\omega\varepsilon)$.

The simple rule of thumb is that approximations for imperfect dielectric can be applied when

$$\frac{\sigma}{\omega\varepsilon} \leq 0.1$$

When the condition above is verified, the imperfect dielectric behaves in all respects like a perfect dielectric, except for an attenuation term in the fields.

The quantity $\sigma/\omega\varepsilon$ is called Loss Tangent.

Good conductors – For these materials $\sigma \neq 0$ but $(\sigma/\omega\varepsilon) >> 1$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} \approx \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\sqrt{j}$$

$$= \sqrt{\omega\mu\sigma}\,\exp(j\frac{\pi}{4}) = \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\pi f\mu\sigma}(1 + j)$$

$$\alpha \approx \sqrt{\pi f\mu\sigma} \qquad \beta \approx \sqrt{\pi f\mu\sigma}$$

$$v_p = \frac{\omega}{\beta} \approx \sqrt{\frac{4\pi f}{\mu\sigma}} \qquad \lambda = \frac{2\pi}{\beta} \approx \sqrt{\frac{4\pi}{f\mu\sigma}}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\exp(j\frac{\pi}{4})$$

$$= \sqrt{\frac{\omega\mu}{\sigma}}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{\pi f\mu}{\sigma}}(1 + j)$$

The simple rule of thumb is that approximations for good conductor can be applied when

$$\frac{\sigma}{\omega\varepsilon} \geq 10$$

Note that for a good conductor the attenuation constant α and the propagation constant β are approximately equal.

The medium impedance η has nearly equal real and imaginary parts, therefore its phase angle is approximately 45°.

This means that in a good conductor the electric and magnetic fields have always a phase difference $\tau = 45° = \pi/4$.

Also, in a good conductor the fields attenuate very rapidly. The distance over which fields are attenuated by a factor exp(−1.0) is

$$\frac{1}{\alpha} = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \textbf{Skin depth}$$

A typical good conductor is copper, which has the following parameters:

$$\sigma = 5.80 \times 10^{7} \, [\text{S/m}]$$

$$\varepsilon \approx \varepsilon_o$$

$$\mu \approx \mu_o$$

Copper remains a good conductor at extremely high frequencies.
Another good conductor example is sea water at relatively low frequencies

$$\sigma \approx 4.0 \text{ [S/m]}$$

$$\varepsilon \approx 80\varepsilon_o$$

$$\mu \approx \mu_o$$

At a frequency of 25 kHz

$$\frac{\sigma}{\omega\varepsilon} \approx 36,000$$

Perfect conductor - For this ideal material $\sigma \rightarrow \infty$

For this material, the attenuation is also infinite and the skin depth goes to zero. This means that the electromagnetic field must go to zero below the perfect conductor surface.

General medium - When a material is not covered by one of the limit cases, the complete formulation must be used. We can classify a material for which the conditions $(\sigma/\omega\varepsilon) \ll 1$ or $(\sigma/\omega\varepsilon) \gg 10$ are invalid as a general medium.

The simple rule of thumb for general medium is

$$10 > \frac{\sigma}{\omega\varepsilon} > 0.1$$