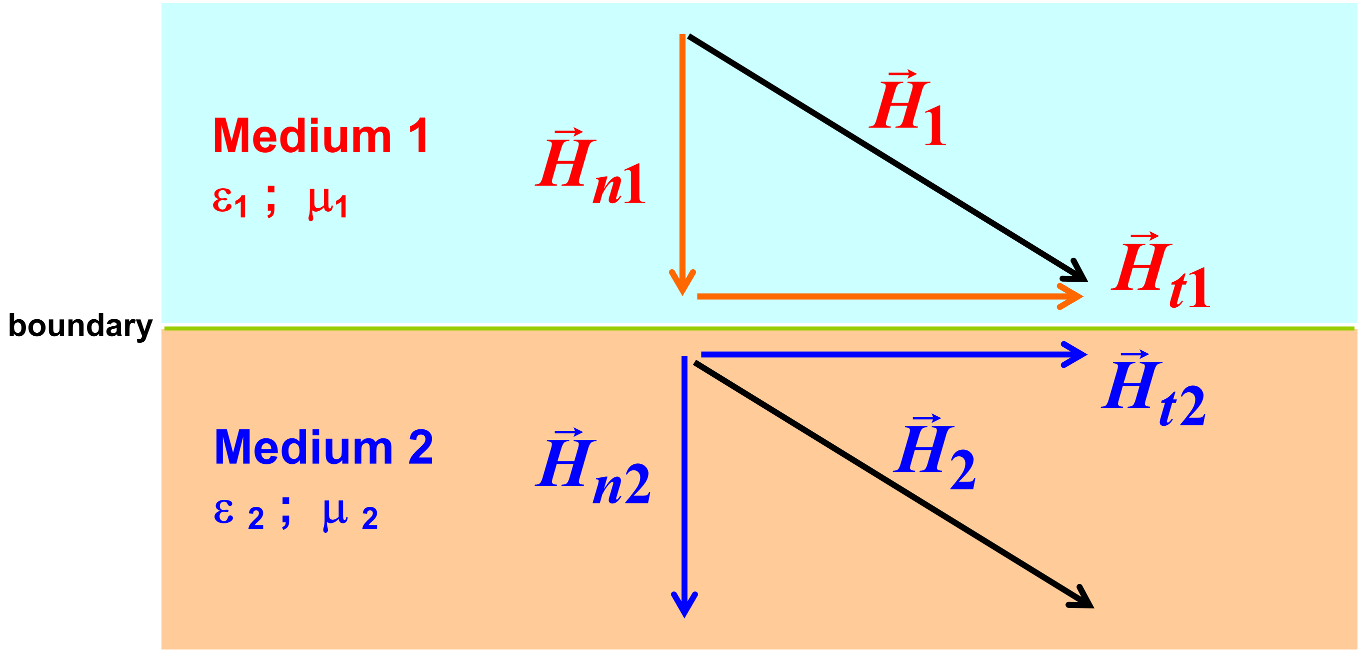
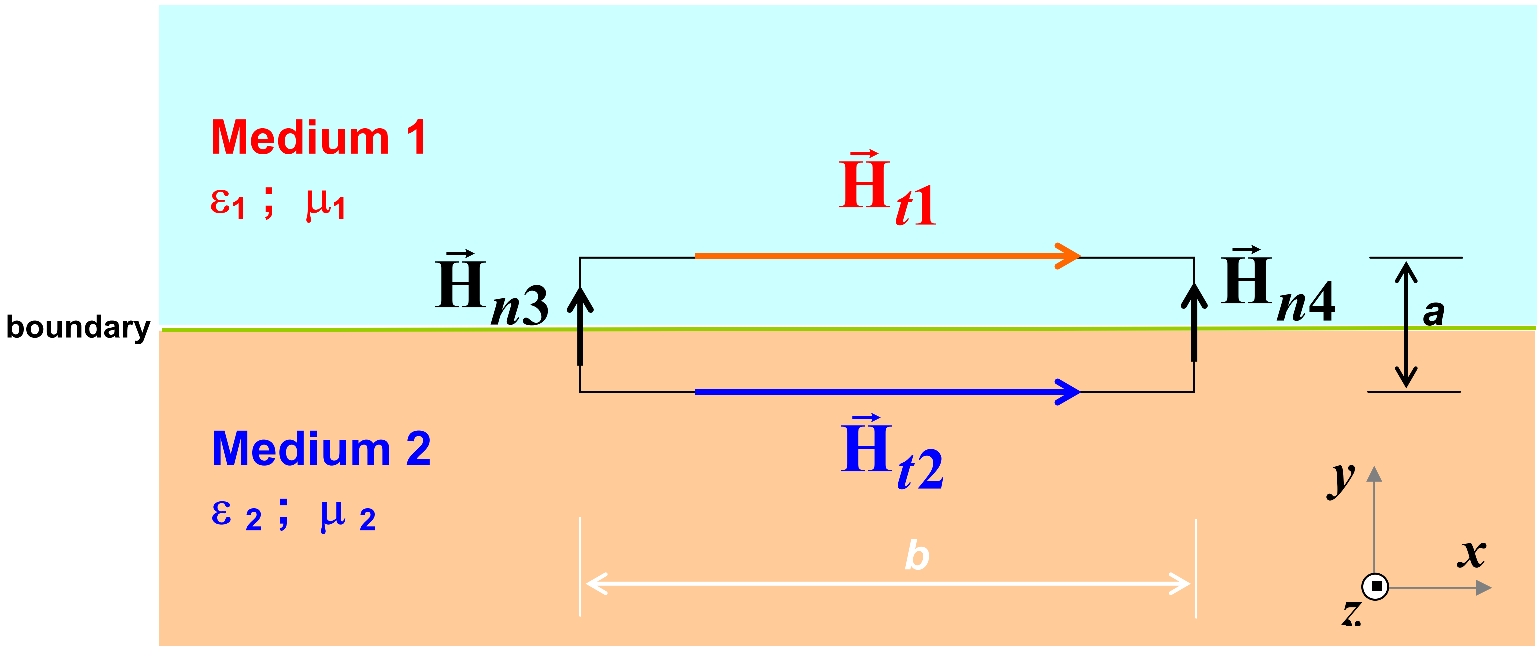
Review of Boundary Conditions

Consider an electromagnetic field at the boundary between two materials with different properties. The tangent and the normal component of the fields must be examined separately, in order to understand the effects of the boundary.



Tangential Magnetic Field



Ampère's law for the boundary region in the figure can be written as

$$\nabla \times \vec{\mathbf{H}} \Rightarrow \frac{\partial \mathbf{H}_y}{\partial x} - \frac{\partial \mathbf{H}_x}{\partial y} = J_z + j\omega\,\varepsilon\,\mathbf{E}_z$$

In terms of finite differences approximation for the derivatives

$$\frac{H_{n4} - H_{n3}}{b} - \frac{H_{t1} - H_{t2}}{a} = J_z + j\omega\,\varepsilon E_z$$

If one lets the boundary region shrink, with a **going to zero faster than** b**,**

$$H_{t2} - H_{t1} = \lim_{a \to 0} \left(J_z\, a + j\omega\varepsilon\, E_z a + a \frac{H_{n3} - H_{n4}}{b} \right)$$

for materials with finite conductivity

$$\Rightarrow H_{t2} - H_{t1} = 0$$ **Tangential components are conserved**

for perfect conductors

$$\Rightarrow H_{t2} - H_{t1} = \lim_{a \to 0} (J_z a) = J_s \quad \text{(surface current)}$$

For a general boundary geometry

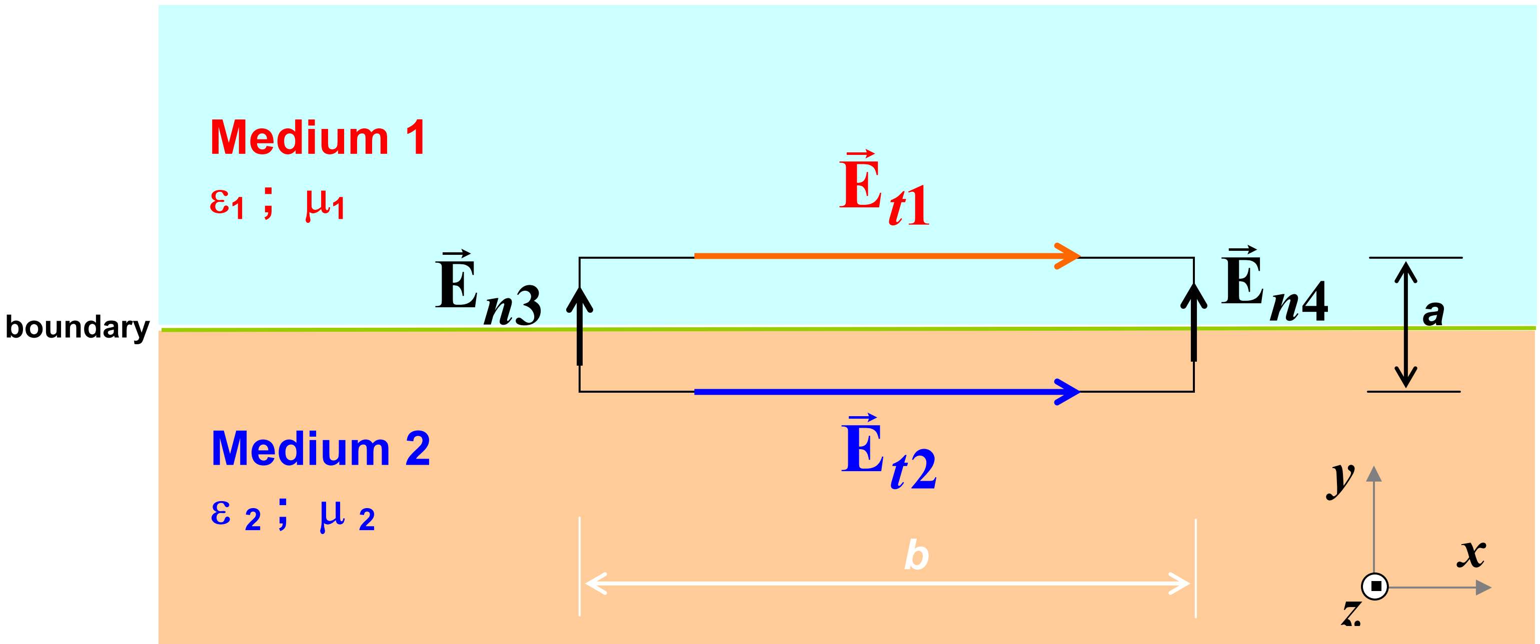
$$\hat{n} \times (\vec{H}_{t1} - \vec{H}_{t2}) = \vec{J}_s$$

$\hat{n}$ = **unit vector normal to the surface**

In the case of a perfect conductor, the electromagnetic fields go immediately to zero inside the material, because the conductivity is infinite and attenuates instantly the fields. The surface current is confined to an infinitesimally thin "skin", and it accounts for the discontinuity of the tangential magnetic field, which becomes immediately zero inside the perfect conductor.

For a real medium, with finite conductivity, the fields can penetrate over a certain distance, and there is a current distributed on a thin, but not infinitesimal, skin layer. The tangential field components on the two sides of the interface are the same. Nonetheless, the perfect conductor is often a good approximation for a real metal.

Tangential Electric Field



Faraday's law for the same boundary region can be written as

$$\nabla \times \vec{\mathbf{E}} \Rightarrow \frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} = j\omega\mu\mathbf{H}_z$$

In terms of finite differences approximation for the derivatives

$$\frac{E_{n4} - E_{n3}}{b} - \frac{E_{t1} - E_{t2}}{a} = j\omega\mu\, H_z$$

If one lets the boundary region shrink, with a going to zero faster than b,

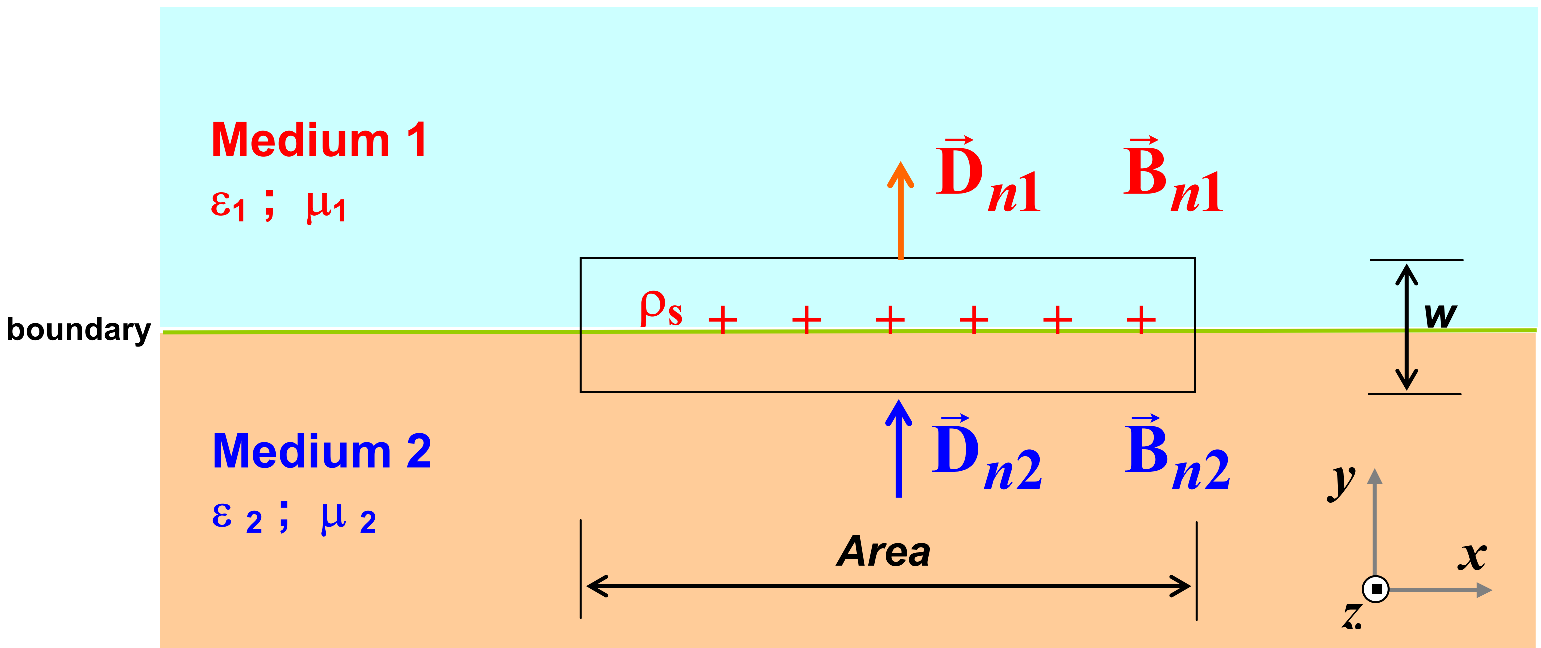
$$E_{t2} - E_{t1} = \lim_{a\to 0} \left(j\omega\mu\, H_z a + a \frac{E_{n3} - E_{n4}}{b} \right)$$

$$\Rightarrow E_{t2} - E_{t1} = 0$$ **Tangential components are conserved**

For a general boundary geometry

$$\hat{n} \times (\vec{E}_{t1} - \vec{E}_{t2}) = 0$$

Normal components



Consider a small box that encloses a certain area of the interface with

$$\rho_s = \textbf{interface charge density}$$

Integrate the divergence of the fields over the volume of the box:

$$\iiint_{Volume} \nabla \cdot \vec{\mathbf{D}}\, d\vec{r} = \iiint_{Volume} \rho\, d\vec{r}$$

Divergence $\Downarrow$ **theorem**

$$\oiint_{Surface} \vec{\mathbf{D}} \cdot \hat{\mathbf{n}}\; d\vec{s} = \text{Flux of } \vec{\mathbf{D}} \text{ out of the box}$$

$$\iiint_{Volume} \nabla \cdot \vec{\mathbf{B}}\, d\vec{r} = 0$$

Divergence $\Downarrow$ **theorem**

$$\oiint_{Surface} \vec{\mathbf{B}} \cdot \hat{\mathbf{n}}\; d\vec{s} = \text{Flux of } \vec{\mathbf{B}} \text{ out of the box}$$

If the thickness of the box tends to zero and the charge density is assumed to be uniform over the area, we have the following fluxes

$$\vec{D}\text{-Flux out of box} = Area \cdot (D_{1n} - D_{2n}) =$$
$$= \text{Total interface charge} = Area \cdot \rho_s$$

$$\vec{B}\text{-Flux out of box} = Area \cdot (B_{1n} - B_{2n}) = 0$$

The resulting boundary conditions are

$$D_{1n} - D_{2n} = \rho_s \qquad\qquad B_{1n} - B_{2n} = 0$$

The discontinuity in the normal component of the displacement field D is equal to the density of surface charge.

The normal components of the magnetic induction field B are continuous across the interface.

For isotropic and uniform values of ε and μ in the two media

$$\vec{\mathbf{D}}_{n1} - \vec{\mathbf{D}}_{n2} = \varepsilon_1 \vec{\mathbf{E}}_{n1} - \varepsilon_2 \vec{\mathbf{E}}_{n2} = \rho_s$$

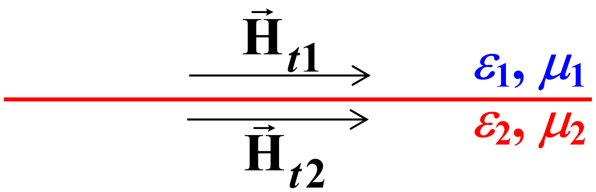
$$\vec{\mathbf{B}}_{n1} - \vec{\mathbf{B}}_{n2} = \mu_1 \vec{\mathbf{H}}_{n1} - \mu_2 \vec{\mathbf{H}}_{n2} = 0$$

Even when the interface charge is zero, the normal components of the electric field are discontinuous at the interface, if there is a change of dielectric constant .

The normal components of the magnetic field have a similar discontinuity at the interface due to the change in the magnetic permeability. In many practical situations, the two media may have the same permeability as vacuum, μ_0, and in such cases the normal component of the magnetic field is conserved across the interface.

SUMMARY

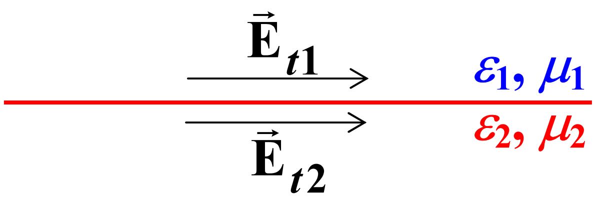
If medium 2 is perfect conductor



$$\vec{H}_{t1} = \vec{H}_{t2}$$

$$\hat{n} \times \vec{H}_{t1} = \vec{J}_s$$

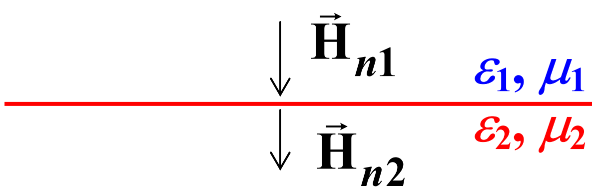
$$\vec{H}_{t2} = 0$$



$$\vec{E}_{t1} = \vec{E}_{t2}$$

$$\vec{E}_{t1} = 0$$

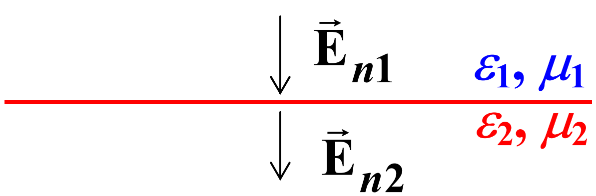
$$\vec{E}_{t2} = 0$$



$$\mu_1 \vec{H}_{n1} = \mu_2 \vec{H}_{n2}$$

$$\vec{H}_{n1} = 0$$

$$\vec{H}_{n2} = 0$$



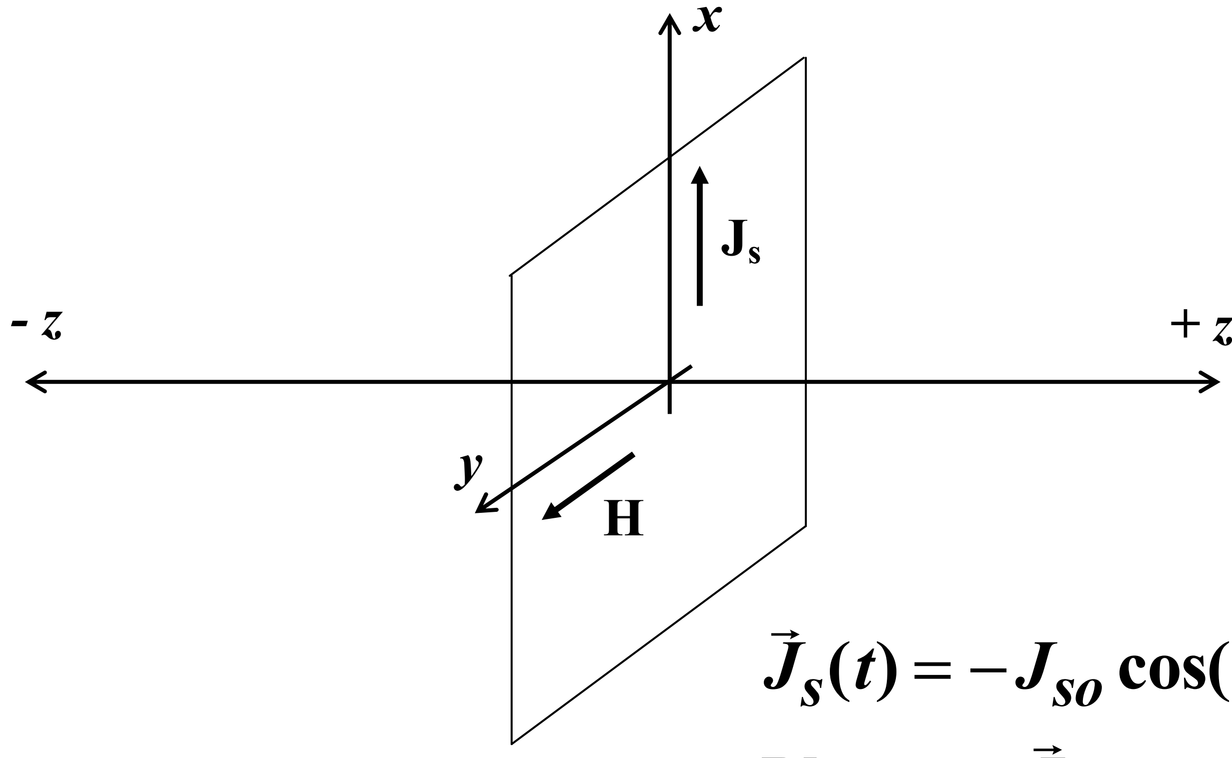
$$\varepsilon_1 \vec{E}_{n1} = \varepsilon_2 \vec{E}_{n2} + \rho_s$$

$$\vec{E}_{n1} = \rho_s / \varepsilon_1$$

$$\vec{E}_{n2} = 0$$

Examples:

An infinite current sheet generates a plane wave (free space on both sides)



$$\vec{J}_s(t) = -J_{so}\cos(\omega t)\,\hat{i}_x$$

$$\text{Phasor} \quad \vec{\mathbf{J}}_s = -J_{so}\,\hat{i}_x$$

The E.M. field is transmitted on both sides of the infinitesimally thin sheet of current.

BOUNDARY CONDITIONS

$$\hat{n} \times (\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2}) = \vec{\mathbf{J}}_s$$

$$\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2} = J_{so} \hat{i}_x$$

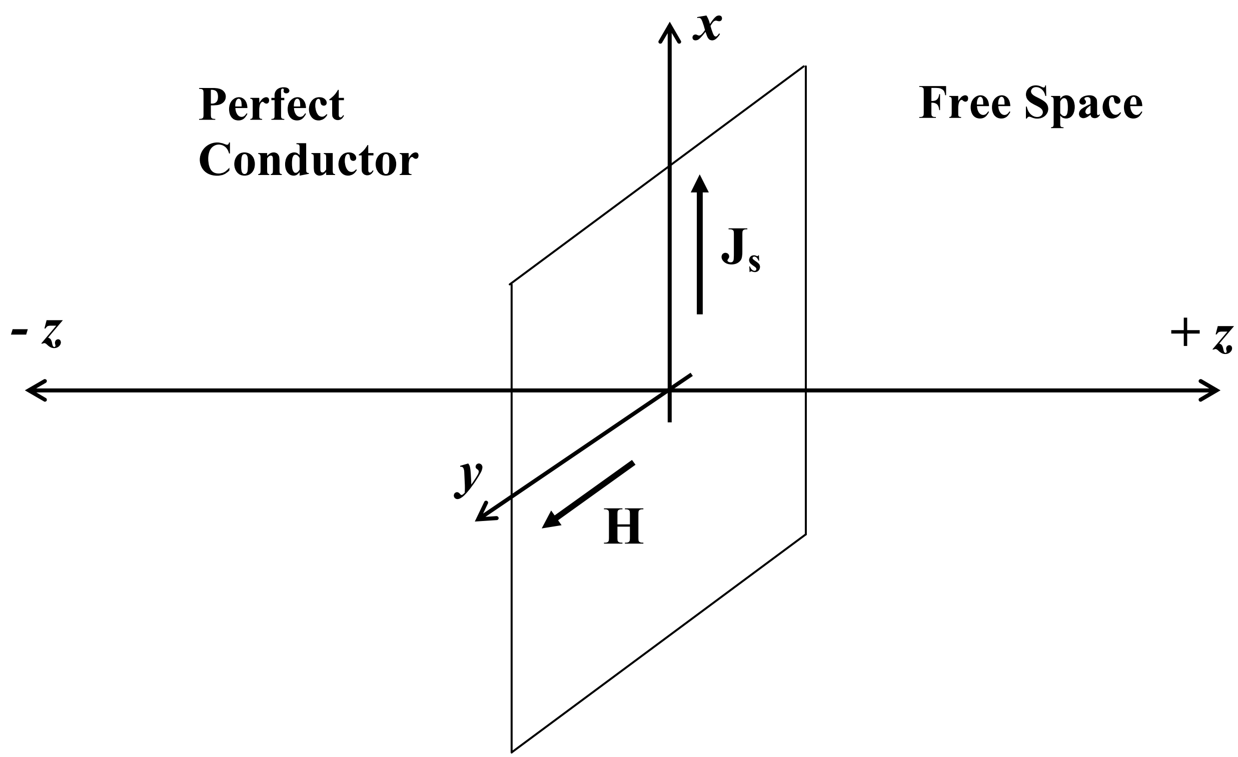
$$\vec{\mathbf{E}}_{t1} = \vec{\mathbf{E}}_{t2}$$

$$\left| \vec{\mathbf{E}}_{t1} \right| = \eta_0 \left| \vec{\mathbf{H}}_{t1} \right|$$

$$\text{Symmetry} \Rightarrow \left| \vec{\mathbf{H}}_{t1} \right| = \left| \vec{\mathbf{H}}_{t2} \right|$$

$$\mathbf{H}_1 = \frac{J_{so}}{2} \qquad \mathbf{H}_2 = -\frac{J_{so}}{2}$$

A semi-infinite perfect conductor medium in contact with free space has uniform surface current and generates a plane wave



$$\vec{\mathbf{J}}_s = -J_{so}\cos(\omega t)\,\hat{i}_x$$

The E.M. field is zero inside the perfect conductor. The wave is only transmitted into free space.

BOUNDARY CONDITIONS

$$\hat{n} \times (\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2}) = \vec{\mathbf{J}}_s$$

$$\vec{\mathbf{H}}_{t1} - \vec{\mathbf{H}}_{t2} = \vec{\mathbf{H}}_{t1} - 0 = J_{so}\, \hat{i}_x$$

$$\vec{\mathbf{E}}_{t2} = 0$$

$$\text{Asymmetry} \Rightarrow \left|\vec{\mathbf{H}}_{t1}\right| \neq \left|\vec{\mathbf{H}}_{t2}\right|$$

$$\mathbf{H}_{t1} = J_{so} \qquad \mathbf{H}_{t2} = 0$$