Electromagnetic Waves

For fast-varying phenomena, the displacement current cannot be neglected, and the full set of Maxwell's equations must be used

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}(t)}{dt}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}(t)}{\partial t}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

The two curl equations are analogous to the coupled (first order) equations for voltage and current used in transmission lines. The solutions of this system of equations are waves. In order to obtain uncoupled (second order) equations we can operate with the curl once more. Under the assumption of uniform isotropic medium:

$$\nabla \times \nabla \times \vec{E}(t) = -\frac{\partial\left(\nabla \times \vec{B}(t)\right)}{\partial t} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}(t)$$

$$= -\mu \frac{\partial \vec{J}(t)}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2}$$

$$\nabla \times \nabla \times \vec{H}(t) = \nabla \times \vec{J} + \frac{\partial\left(\nabla \times \vec{D}(t)\right)}{\partial t} = \nabla \times \vec{J} + \varepsilon \frac{\partial}{\partial t} \nabla \times \vec{E}(t)$$

$$= \nabla \times \vec{J} - \varepsilon\mu \frac{\partial^2 \vec{H}(t)}{\partial t^2}$$

From vector calculus, we also have

$$\nabla\times\nabla\times\vec{E}(t)=\nabla\nabla\cdot\vec{E}(t)-\nabla^2\vec{E}(t)$$

$$\nabla\times\nabla\times\vec{H}(t)=\nabla\underbrace{\nabla\cdot\vec{H}(t)}-\nabla^2\vec{H}(t)=-\nabla^2\vec{H}(t)$$

$$\boxed{\frac{1}{\mu}\nabla\cdot\vec{B}(t)=0}$$

Finally, we obtain the general wave equations

$$\nabla^2\vec{E}(t)-\nabla\nabla\cdot\vec{E}(t)-\mu\,\varepsilon\frac{\partial^2\vec{E}(t)}{\partial t^2}=\mu\frac{\partial\vec{J}(t)}{\partial t}$$

$$\nabla^2\vec{H}(t)-\mu\,\varepsilon\frac{\partial^2\vec{H}(t)}{\partial t}=-\nabla\times\vec{J}(t)$$

In a region where the wave solution propagates away from charges and flowing currents, the wave equations can be simplified considerably. In such conditions, we have

$$\rho = 0 \quad \Rightarrow \quad \nabla \cdot \vec{E}(t) = \rho / \varepsilon = 0$$

$$\vec{J}(t) = 0$$

and the wave equations assume the familiar form

$$\nabla^2 \vec{E}(t) - \mu \varepsilon \frac{\partial^2 \vec{E}(t)}{\partial t^2} = 0$$

$$\nabla^2 \vec{H}(t) - \mu \varepsilon \frac{\partial^2 \vec{H}(t)}{\partial t} = 0$$

When currents and charges are involved, the wave equations are difficult to solve, because of the terms

$$\nabla\left(\nabla\cdot\vec{E}(t)\right) \qquad \text{and} \qquad \nabla\times\vec{J}(t)$$

It is more practical to have equations for the electric potential and for the magnetic vector potential, which contain linear source terms dependent on charge and current, as shown below.

We saw earlier that the divergence of the magnetic vector potential must be specified. The simple choice made in magnetostatics of zero divergence is not suitable for time-varying fields. Among the possible choices, it is convenient to adopt the Lorenz gauge

Time-varying fields – Lorenz gauge	Magnetostatics (d.c.)
$\nabla\cdot\vec{A}(t) = -\mu\,\varepsilon\dfrac{\partial\phi}{\partial t}$	$\nabla\cdot\vec{A} = 0$

Starting from the definitions

$$\vec{B}(t) = \nabla \times \vec{A} \qquad \vec{E}(t) = -\frac{\partial \vec{A}(t)}{\partial t} - \nabla \phi$$

we obtain again the wave equation by applying the curl operation

$$\mu \nabla \times \vec{H}(t) = \nabla \times \nabla \times \vec{A}(t) = \nabla \nabla \cdot \vec{A}(t) - \nabla^2 \vec{A}(t) =$$

$$= \mu \vec{J} - \varepsilon \mu \frac{\partial^2 \vec{A}(t)}{\partial t^2} \boxed{- \varepsilon \mu \nabla \frac{\partial \phi}{\partial t}}$$

$$\boxed{= \nabla \nabla \cdot \vec{A}(t)}$$

With the application of Lorenz gauge

$$\boxed{\nabla^2 \vec{A}(t) - \varepsilon \mu \frac{\partial^2 \vec{A}(t)}{\partial t^2} = -\mu \vec{J}(t)}$$

For the electric potential we have

$$\nabla \cdot \vec{D}(t) = \rho \qquad \Rightarrow \qquad \nabla \cdot \vec{E}(t) = \frac{\rho}{\varepsilon}$$

$$\nabla \cdot \vec{E}(t) = \nabla \cdot \left(-\frac{\partial \vec{A}(t)}{\partial t} - \nabla \phi \right) = \frac{\rho}{\varepsilon}$$

$$-\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \vec{A}(t) = \frac{\rho}{\varepsilon}$$

After applying the Lorenz gauge once more, we arrive at the potential wave equation

$$\nabla^2 \phi - \varepsilon \mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

In engineering it is very important to consider time-harmonic fields with a sinusoidal time-variation. If we assume a steady-state situation (after all transients have died out) most physical situations may be investigated by considering one single frequency at a time.

This assumption leads to great simplifications in the algebra. It is also realistic, because in practical electromagnetics applications we often have a dominant frequency (carrier) to consider.

The time-harmonic fields have the form

$$\vec{E}(t) = \vec{E}_0 \cos(\omega t + \varphi_E) \qquad \vec{H}(t) = \vec{H}_0 \cos(\omega t + \varphi_H)$$

We can use the complex phasor representation

$$\vec{E}(t) = \mathrm{Re}\left\{ \vec{E}_0 e^{j\varphi_E} e^{j\omega t} \right\} \qquad \vec{H}(t) = \mathrm{Re}\left\{ \vec{H}_0 e^{j\varphi_H} e^{j\omega t} \right\}$$

We define

$$\vec{\mathrm{E}} = \vec{E}_0 e^{j\varphi_E} = \text{phasor of } \vec{E}(t)$$

$$\vec{\mathrm{H}} = \vec{H}_0 e^{j\varphi_H} = \text{phasor of } \vec{H}(t)$$

Maxwell's equations can be rewritten for phasors, with the time-derivatives transformed into linear terms

$$j\omega\vec{\mathrm{E}} = \text{phasor of } \frac{\partial \vec{E}(t)}{\partial t}$$

$$-\omega^2\vec{\mathrm{E}} = \text{phasor of } \frac{\partial^2 \vec{E}(t)}{\partial t^2}$$

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In phasor form, Maxwell's equations become

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}}$$

$$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + j\omega\varepsilon\,\vec{\mathbf{E}}$$

$$\nabla \cdot \vec{\mathbf{D}} = \rho$$

$$\nabla \cdot \vec{\mathbf{B}} = 0$$

$$\vec{\mathbf{D}} = \varepsilon\,\vec{\mathbf{E}}$$

$$\vec{\mathbf{B}} = \mu\,\vec{\mathbf{H}}$$

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \vec{v} \times \vec{\mathbf{B}}\right)$$

where all electromagnetic quantities are phasors and functions only of space coordinates.

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Let's consider first vacuum as a medium. The wave equations for phasors become Helmholtz equations

$$\nabla^2 \vec{E} + \omega^2 \mu_0 \varepsilon_0 \vec{E} = 0$$

$$\nabla^2 \vec{H} + \omega^2 \mu_0 \varepsilon_0 \vec{H} = 0$$

The general solutions for these differential equations are waves moving in 3-D space. Note, once again, that the two equations are uncoupled.

This means that each equation contains all the necessary information for the total electromagnetic field and one only needs to solve the equation for one field to completely specify the problem. The other field is obtained with a curl operation by invoking one of the original Maxwell equations.

At this stage we assume that a wave exists, and we do not yet concern ourselves with the way the wave is generated. So, for the sake of understanding wave behavior, we can restrict the Helmhlotz equations to a simple case:

- **We assume that the wave solution has an electric field which is uniform on the $\{x, y\}$-plane and has a reference positive orientation along the x-direction. Then, we verify that this is a reasonable choice corresponding to an actual solution of the Helmholtz wave equations. We recall that the Laplacian of a scalar is a scalar**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and that the Laplacian of a vector is a vector

$$\nabla^2 \vec{\mathrm{E}} = \hat{i}_x \nabla^2 \mathrm{E}_x + \hat{i}_y \nabla^2 \mathrm{E}_y + \hat{i}_z \nabla^2 \mathrm{E}_z$$

The Helmholtz equation becomes:

$$\nabla^2 \vec{\mathbf{E}} + \omega^2 \mu_0 \varepsilon_0 \vec{\mathbf{E}} = \frac{\partial^2 \mathbf{E}_x}{\partial z^2} \hat{i}_x + \omega^2 \mu_0 \varepsilon_0 \left(\mathbf{E}_x \hat{i}_x \right) = 0$$

Only the x-component of the electric field exists (due to the chosen orientation) and only the z-derivative exists, because the field is uniform on the $\{ x , y \}$-plane.

We have now a one-dimensional wave propagation problem described by the scalar differential equation

$$\frac{\partial^2 \mathbf{E}_x}{\partial z^2} + \omega^2 \mu_0 \varepsilon_0 \mathbf{E}_x = 0$$

This equation has a well known general solution

$$A \exp(-j\beta z) + B \exp(j\beta z)$$

where the propagation constant is

$$\beta = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c}$$

The wave that we have assumed is a plane wave and we have verified that it is a solution of Helmholtz equation. The general solution above has two possible components

$A \exp(-j\beta z)$ **Forward wave, moving along positive** z

$B \exp(j\beta z)$ **Backward wave, moving along negative** z

For the simple wave orientation chosen here, the problem is mathematically identical to the one solved earlier for voltage propagation in a homogeneous transmission line.

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If a specific electromagnetic wave is established in an infinite homogeneous medium, moving for instance along the positive direction, only the forward wave should be considered.

A reflected wave exists when a discontinuity takes place along the path of the forward wave (that is, the material medium changes properties, either abruprtly or gradually).

We can also assume that the amplitude of the forward plane wave solution is given and that it is in general a complex constant fixed by the conditions that generated the wave

$$A = E_0 \, e^{j\varphi}$$

We can write at last the phasor electric field describing a simple forward plane wave solution of Helmholtz equation as:

$$\vec{\mathrm{E}}_x(z) = E_0 \, e^{j\varphi} \, e^{-j\beta z} \, \hat{i}_x$$

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The corresponding time-dependent field is obtained by applying the inverse phasor transformation

$$\vec{E}_x(z,t) = \mathrm{Re}\left\{ \mathrm{E}_x(z)\, e^{j\omega t}\, \hat{i}_x \right\} = \mathrm{Re}\left\{ E_0\, e^{j\varphi}\, e^{-j\beta z}\, e^{j\omega t}\, \hat{i}_x \right\}$$

$$= E_0 \cos(\omega t - \beta z + \varphi)\, \hat{i}_x$$

The phasor magnetic field is obtained directly from the Maxwell equation for the electric field curl

$$\nabla \times \vec{\mathrm{E}} = \nabla \times \left(E_0\, e^{j\varphi}\, e^{-j\beta z}\, \hat{i}_x \right) = -j\omega\mu_0\, \vec{\mathrm{H}}$$

$$\vec{\mathrm{H}} = -\frac{\nabla \times \left(E_0\, e^{j\varphi}\, e^{-j\beta z}\, \hat{i}_x \right)}{j\omega\mu_0}$$

We then develop the curl as

$$\nabla \times \Big(\underbrace{E_0\, e^{j\varphi}\, e^{-j\beta z}}_{\mathbf{E}_x(z)}\, \hat{i}_x \Big) = \mathbf{det} \begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{E}_x(z) & \mathbf{0} & \mathbf{0} \end{bmatrix} =$$

$$= \frac{\partial \left(E_0\, e^{j\varphi}\, e^{-j\beta z} \right)}{\partial z}\, \hat{i}_y - \underbrace{\frac{\partial \left(E_0\, e^{j\varphi}\, e^{-j\beta z} \right)}{\partial y}}_{=\,\mathbf{0}}\, \hat{i}_z =$$

$$= -j\beta\, E_0\, e^{j\varphi}\, e^{-j\beta z}\, \hat{i}_y$$

The final result for the phasor magnetic field is

$$\vec{\mathbf{H}}_y(z) = -\frac{-j\beta E_0\, e^{j\varphi}\, e^{-j\beta z}}{j\omega\mu}\,\hat{\boldsymbol{i}}_y =$$

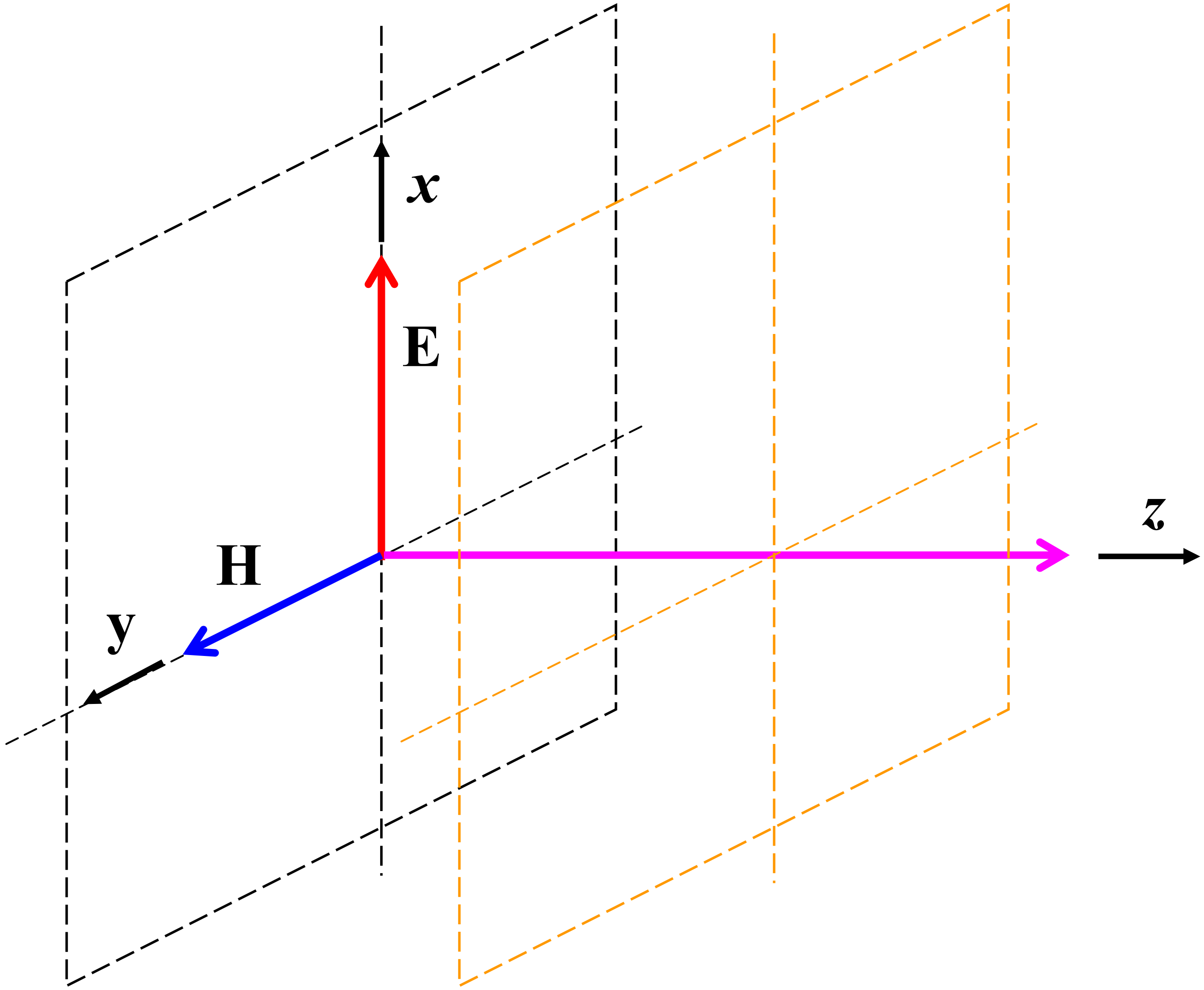
$$= \frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\,\mu_0} E_0\, e^{j\varphi}\, e^{-j\beta z}\,\hat{\boldsymbol{i}}_y =$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_0\, e^{j\varphi}\, e^{-j\beta z}\,\hat{\boldsymbol{i}}_y = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\mathbf{E}_x(z)\,\hat{\boldsymbol{i}}_y$$

We define

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = \eta_0 \approx 377\,\Omega \quad = \quad \text{Intrinsic impedance of vacuum}$$

We have found that the fields of the electromagnetic wave are perpendicular to each other, and that they are also perpendicular (or transverse) to the direction of propagation.



Electromagnetic power flows with the wave along the direction of propagation and it is also constant on the phase-planes. The power density is described by the time-dependent Poynting vector

$$\vec{P}(t) = \vec{E}(t) \times \vec{H}(t)$$

The Poynting vector is perpendicular to both field components, and is parallel to the direction of wave propagation.

When the wave propagates on a general direction, which does not coincide with one of the cartesian axes, the propagation constant must be considered to be a vector with amplitude

$$|\vec{\beta}| = \omega\sqrt{\mu\,\varepsilon}$$

and direction parallel to the Poynting vector.

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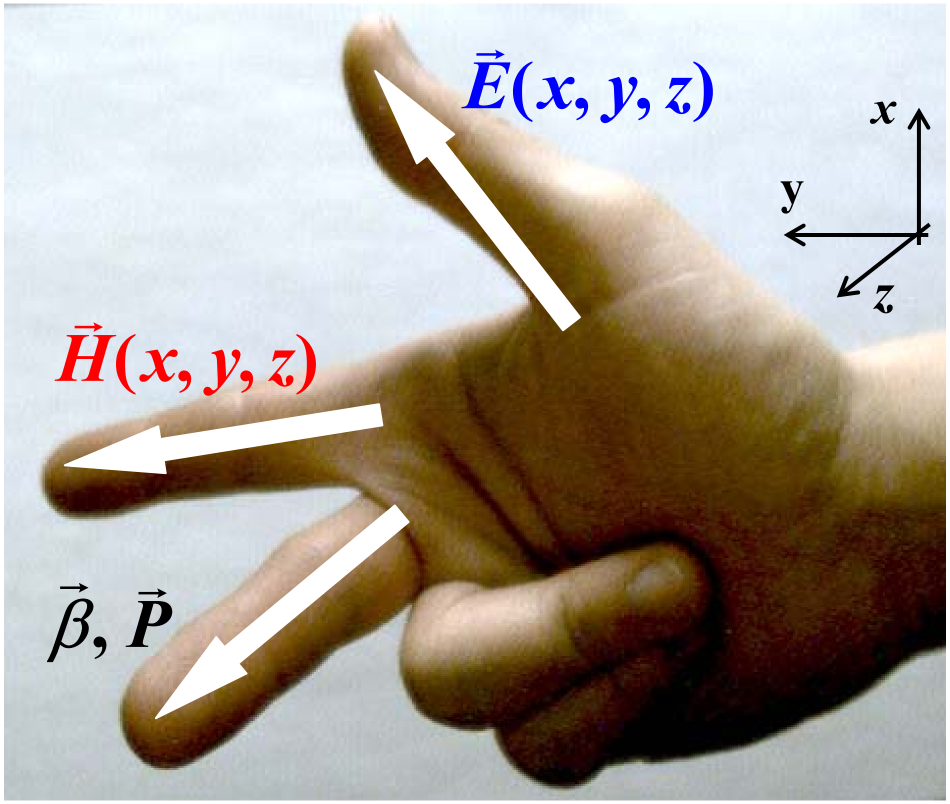
The condition of mutual orthogonality between the field components and the Poynting vector is general and it applies to any plane wave with arbitrary direction of propagation. The mutual orientation chosen for the reference directions of the fields follows the right hand rule.



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So far, we have just verified that electromagnetic plane waves are possible solutions of the Maxwell equations for time-varying fields. One may wonder at this point if plane waves have practical physical relevance.

First of all, we should notice that plane waves are mathematically analogous to the exponential basis functions used in Fourier analysis. This means that a general wave, with more than one frequency component, can always be decomposed in terms of plane waves.

- **For periodic signals, we have a discrete set of waves which are harmonics of the fundamental frequency (analogy with Fourier series).**

- **For general signals, we must consider a continuum of frequencies in order to decompose in terms of elementary plane waves (analogy with Fourier transform).**

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From a physical point of view, however, the properties of a plane wave may be somewhat puzzling.

Assume that a steady-state plane wave is established in an ideal infinite homogeneous medium. On any plane perpendicular to the direction of propagation (phase-planes), the electric and magnetic fields have uniform magnitude and phase.

The electromagnetic power, flowing with a phase-plane of the wave, is obtained by integrating the Poynting vector, which is also uniform on each phase-plane. For a plane where the Poynting vector is non-zero, the total power carried by the wave is infinite

$$\int_{plane} \vec{P}(t) = \int_{plane} \vec{E}(t) \times \vec{H}(t) \rightarrow \infty$$

In many practical cases, we approximate an actual wave with a plane wave on a limited region of space, thus considering an appropriate finite power.