Review: Electrostatics and Magnetostatics

In the static regime, electromagnetic quantities do not vary as a function of time. We have two main cases:

ELECTROSTATICS – The electric charges do not change postion in time. Therefore, ρ, $\boldsymbol{E}$ and $\boldsymbol{D}$ are constant and there is no magnetic field $\boldsymbol{H}$, since there is no current density $\boldsymbol{J}$.

MAGNETOSTATICS – The charge crossing a given cross-section (current) does not vary in time. Therefore, $\boldsymbol{J}$, $\boldsymbol{H}$ and $\boldsymbol{B}$ are constant. Although charges are moving, the steady current maintains a constant charge density ρ in space and the electric field $\boldsymbol{E}$ is static.

The equations of electrostatics are obtained directly from Maxwell's equations, by assuming that $\partial/\partial t$, J, H and B are all zero:

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{D} = 0$$

$$\vec{D} = \varepsilon \, \vec{E}$$

The electrostatic force is simply

$$\vec{F} = q \, \vec{E}$$

We also define the electrostatic potential ϕ by the relationship

$$\vec{E} = -\nabla \phi$$

The electrostatic potential ϕ is a scalar function of space. From vector calculus we know that

$$\nabla \times \nabla \phi = \mathbf{0}$$

This potential is very convenient for practical applications because it is a scalar quantity. The potential automatically satisfies Maxwell's curl equation for the electric field, since

$$\nabla \times \vec{E} = -\nabla \times \nabla \phi = \mathbf{0}$$

From a physical point of view, the electrostatic potential provides an immediate way to express the work W performed by moving a charge from location a to location b:

$$W = -\int_a^b \vec{F} \cdot d\vec{l}$$

Here, l is the coordinate along the path. The negative sign indicates that the work is done against the electrical force.

By introducing the electrostatic potential, we obtain

$$W = -q\int_a^b \vec{E}\cdot d\vec{l} = q\int_a^b \nabla\phi\cdot d\vec{l} = q\left[\phi(b)-\phi(a)\right] = q\,\delta\phi$$

NOTE: The line integral of a gradient does not depend on the path of integration but only on the potential at the end points of the path.

The electrostatic potential is measured with respect to an arbitrary reference value. We can assume for most problems that a convenient reference is a zero potential at an infinite distance. In the result above for the electrostatic work, we could set a zero potential at the initial point of the path, so that $\phi(a)=0$. Either choice of potential reference would give the same potential difference $\delta\phi$.

It is quite convenient to express also the divergence equation in terms of the electrostatic potential. If we assume a uniform material medium:

$$\nabla \cdot \vec{D} = \varepsilon\, \nabla \cdot \vec{E} = -\varepsilon\, \nabla \cdot \nabla\phi = -\varepsilon\, \nabla^2 \phi = \rho$$

This result yields the well known Poisson equation

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon}$$

In the case of $\rho = 0$, we have the classic Laplace equation

$$\nabla^2 \phi = 0$$

If the problem involves a non-uniform medium with varying dielectric permittivity, a more general form of Poisson equation must be used

$$\nabla \cdot \left(\varepsilon \nabla \phi \right) = -\rho$$

Another important equation is obtained by integrating the divergence over a certain volume V

$$\int_V \nabla \cdot \vec{D}\ dV = \int_V \rho\ dV$$

Gauss theorem allows us to transform the volume integral of the divergence into a surface integral of the flux

$$\int_V \nabla \cdot \vec{D}\, dV = \int_S \underbrace{\vec{D} \cdot d\vec{S}}$$

Component normal to the surface

The volume integral of the charge density is simply the total charge Q contained inside the volume

$$\int_V \rho \, dV = Q$$

The final result is the integral form of Poisson equation, known as Gauss law:

$$\int_S \vec{D} \cdot d\vec{S} = Q$$

Most electrostatic problems can be solved by direct application of Poisson equation or of Gauss law.

Analytical solutions are usually possible only for simplified geometries and charge distributions, and numerical solutions are necessary for most general problems.

The Gauss law provides a direct way to determine the force between charges. Let's consider a sphere with radius r surrounding a charge Q_1 located at the center. The displacement vector will be uniform and radially directed, anywhere on the sphere surface, so that

$$\int_S \vec{D} \cdot d\vec{S} = 4\pi\, r^2 \left|\vec{D}\right| = Q_1$$

Assuming a uniform isotropic medium, we have a radial electric field with strength

$$\left|\vec{E}(r)\right| = \frac{\left|\vec{D}\right|}{\varepsilon} = \frac{Q_1}{4\,\pi\,\varepsilon\, r^2}$$

If a second charge Q_2 is placed at distance r from Q_1 the mutual force has strength

$$\left|\vec{F}\right| = Q_2 \left|\vec{E}(r)\right| = \frac{Q_1\, Q_2}{4\,\pi\,\varepsilon\, r^2}$$

The electrostatic potential due to a charge Q can be obtained using the previous result for the electrostatic work:

$$\phi(b) = \phi(a) - \int_a^b \left|\vec{E}\right| dr = \phi(a) - \int_a^b \frac{Q}{4\pi\varepsilon r^2}\, dr$$

$$= \phi(a) + \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

where r indicates the distance of the *observation* point b from the charge location, and a is a reference point. If we chose the <u>reference point</u> $a \to \infty$, with a <u>reference potential</u> $\phi(a) = 0$, we can express the potential at distance r from the charge Q as

$$\phi(r) = \frac{Q}{4\pi\varepsilon r}$$

The potential ϕ indicates then the work necessary to move an infinitesimal positive probe charge from distance r (point b) to infinity (point a) for negative Q, or conversely to move the probe from infinity to distance r for positive Q (remember that the work is done against the field). The probe charge should be infinitesimal, not to perturb the potential established by the charge Q.

The work per unit charge done by the fields to move a probe charge between two points, is usually called Electromotive Force (emf). Dimensionally, the emf really represents work rather than an actual force.

The work per unit charge done against the fields represents the voltage V_{ba} between the two points, so that:

$$emf = \int_a^b \vec{E} \cdot d\,\vec{l} = -V_{ba}$$

In electrostatics, there is no difference between voltage and potential. To summarize once again, using formulas, we have

Potential at point " *a* "

$$\phi_a = -\int_{\infty}^{a} \vec{E} \cdot d\,\vec{l}$$

Potential at point " *b* "

$$\phi_b = -\int_{\infty}^{b} \vec{E} \cdot d\,\vec{l}$$

Potential difference or Voltage between " *a* " and " *b*

$$\left(\phi_b - \phi_a\right) = V_{ba} = -\int_{a}^{b} \vec{E} \cdot d\,\vec{l} = -e.m.f.$$

In the case of more than one point charge, the separate potentials due to each charge can be added to obtain the total potential

$$\phi(x, y, z) = \frac{1}{4\,\pi\,\varepsilon} \sum_i \frac{Q_i}{r_i}$$

If the charge is distributed in space with a density $\rho(x,y,z)$, one needs to integrate over the volume as

$$\phi(x, y, z) = \frac{1}{4\,\pi\,\varepsilon} \int_V \frac{\rho(x', y', z')}{r}\, dV$$

In the magnetostatic regime, there are steady currents in the system under consideration, which generate magnetic fields (we ignore at this point the case of ferromagnetic media). The full set of Maxwell's equations is considered (setting $\partial/\partial t = 0$)

$$\nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{D} = \varepsilon \, \vec{E}$$

$$\vec{B} = \mu \, \vec{H}$$

with the complete Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

It is desirable to find also for the magnetic field a potential function. However, note that such a potential cannot be a scalar, as we found for the electrostatic field, since

$$\nabla \times \vec{H} = \vec{J}$$

Current density is a vector ≠ 0

We define a magnetic vector potential *A* through the relation

$$\nabla \times \vec{A} = \vec{B} = \mu \, \vec{H}$$

This definition automatically satisfies the condition of zero divergence for the induction field, since

$$\nabla \cdot \vec{B} = \nabla \cdot \left(\nabla \times \vec{A} \right) = 0$$

The divergence of a curl is always = 0

The vector potential can be introduced in the curl equation for the magnetic field

$$\nabla \times \vec{H} = \frac{1}{\mu} \nabla \times \vec{B} = \frac{1}{\mu} \nabla \times \left(\nabla \times \vec{A} \right) = \frac{1}{\mu} \nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} = \vec{J}$$

However, in order to completely specify the magnetic vector potential, we need to specify also its divergence. First, we observe that the definition of the vector potential is not unique since:

$$\nabla \times \vec{A}' = \nabla \times \left(\vec{A} + \nabla \psi \right) = \nabla \times \vec{A} + \nabla \times \left(\nabla \psi \right) = \nabla \times \vec{A}$$

ψ = Scalar function

Always = 0

In the magnetostatic case it is sufficient to specify (in physics terminology: to choose the gauge)

$$\nabla \cdot \vec{A} = 0$$

so that

$$\nabla \cdot \vec{A} = \nabla \cdot \left(\vec{A} + \nabla \psi \right) = \nabla \cdot \vec{A} + \nabla \cdot \nabla \psi = \nabla \cdot \vec{A} + \nabla^2 \psi = 0$$

We simply need to make sure that the arbitrary function ψ satisfies

$$\nabla^2 \psi = 0$$

We can then simplify the previous result for the curl equation to

$$\boxed{\nabla^2 \vec{A} = -\mu \, \vec{J}}$$

the magnetic equivalent of the electrostatic Poisson equation.

The general solution of this vector Laplacian equation is given by

$$\vec{A}(x,y,z) = \frac{\mu}{4\pi}\int_V \frac{\vec{J}(x',y',z')}{r}\, dV$$

which is similar to the formal solution obtained before for the electrostatic potential for a distributed charge. If the current is confined to a wire with cross-sectional area S and described by a curvilinear coordinate l, we can write

$$\left|\vec{I}\right| = I = \vec{J}\cdot\vec{S} \qquad\qquad dV = \vec{S}\cdot d\vec{l}$$

with a final result

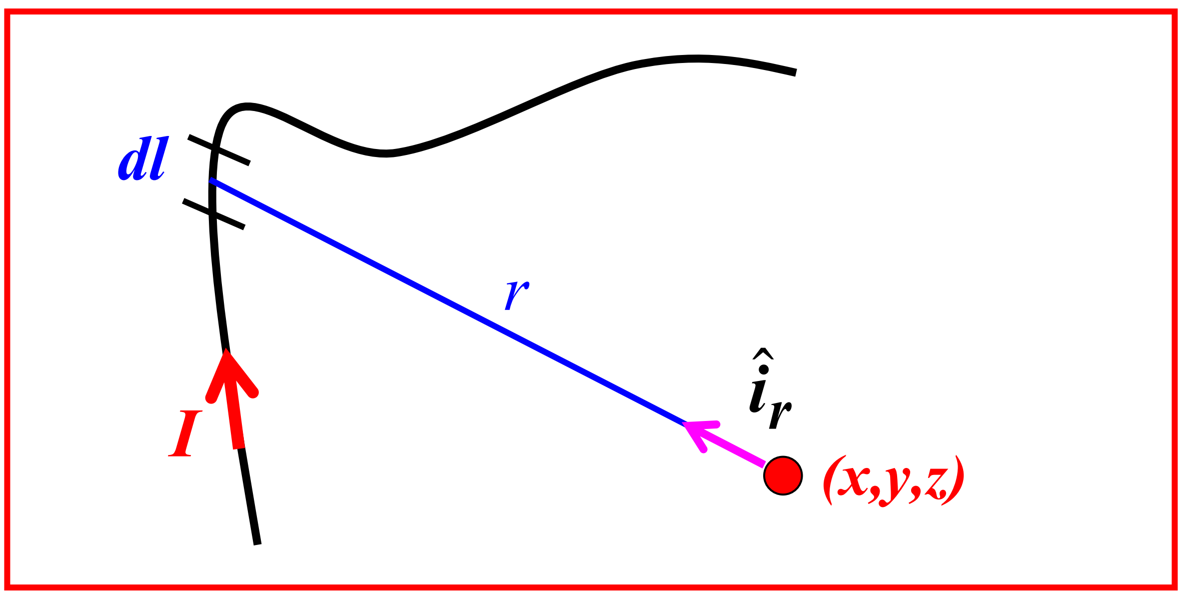
$$\vec{A}(x,y,z) = \frac{\mu I}{4\pi}\int_l \frac{d\vec{l}}{r}$$

(note that the total current I is constant at any wire location).

The solution for the magnetic field obtained from the vector potential leads to the famous Biot-Savart law:

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{1}{\mu} \nabla \times \vec{A} = \nabla \times \left(\frac{I}{4\pi} \int_l \frac{d\vec{l}}{r} \right) = \frac{I}{4\pi} \int_l \nabla \times \frac{d\vec{l}}{r}$$

$$= -\frac{I}{4\pi} \int_l \left(\nabla \frac{1}{r} \right) \times d\vec{l} = \frac{I}{4\pi} \int_l \frac{d\vec{l} \times \hat{i}_r}{r^2}$$



The magnetic field can also be determined by direct integration of the curl equation over a surface

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I$$

Stoke's theorem can be used to transform the left hand side of the equation, to obtain the integral form of Ampere's law

$$\int_l \vec{H} \cdot d\vec{l} = I$$

In many applications it is useful to determine the magnetic flux through a given surface. The vector potential can be used to modify a surface integral into a line integral, using again Stoke's theorem

$$\text{Magnetic Flux} = \int_S \vec{B} \cdot d\vec{S} = \int_S \nabla \times \vec{A} \cdot d\vec{S} = \int_l A \cdot d\vec{l}$$