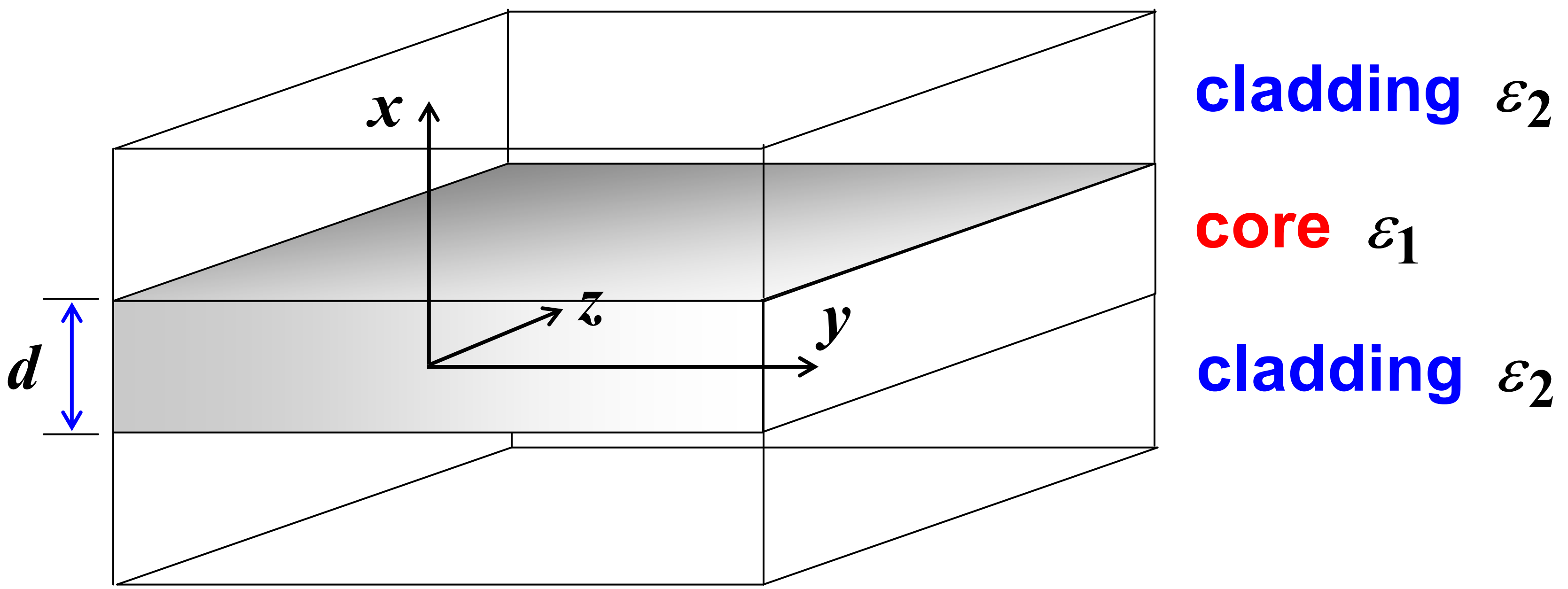
Dielectric Wave Guide

A dielectric waveguide is a structure which exploits total reflection at dielectric interfaces to guide electromagnetic radiation.

The simplest case is the symmetric dielectric slab wave guide

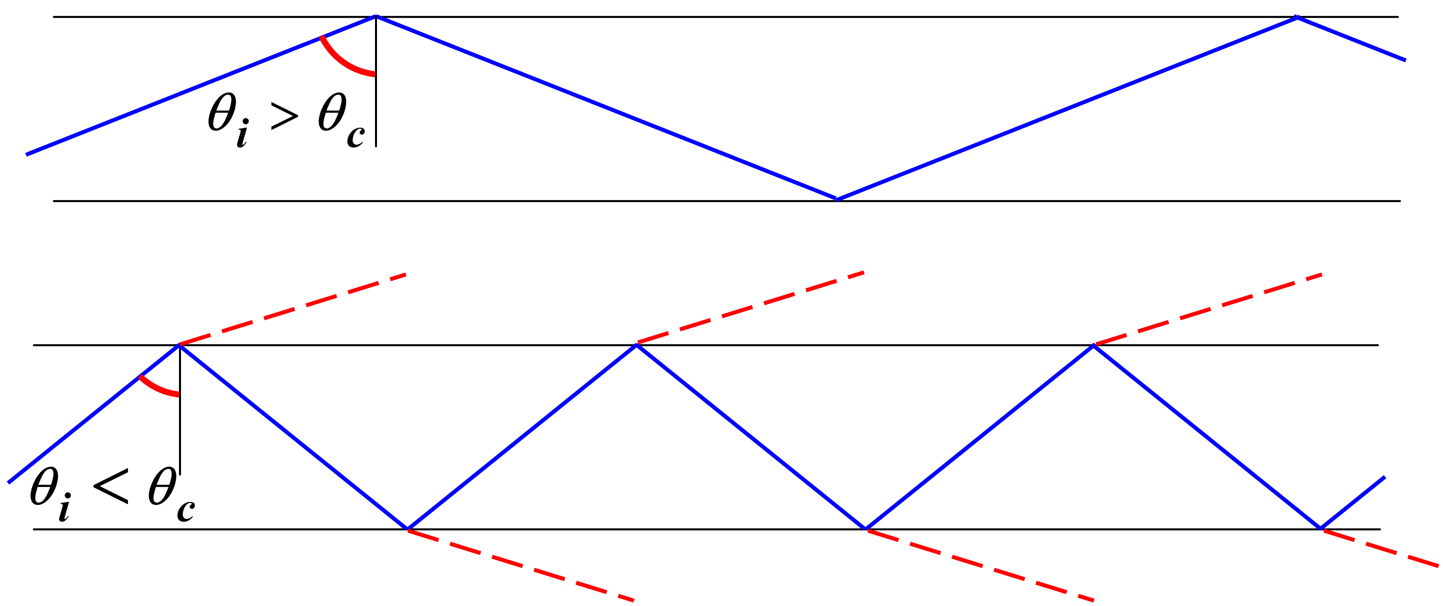


For guidance one must have

$$\boxed{\varepsilon_1 > \varepsilon_2}$$

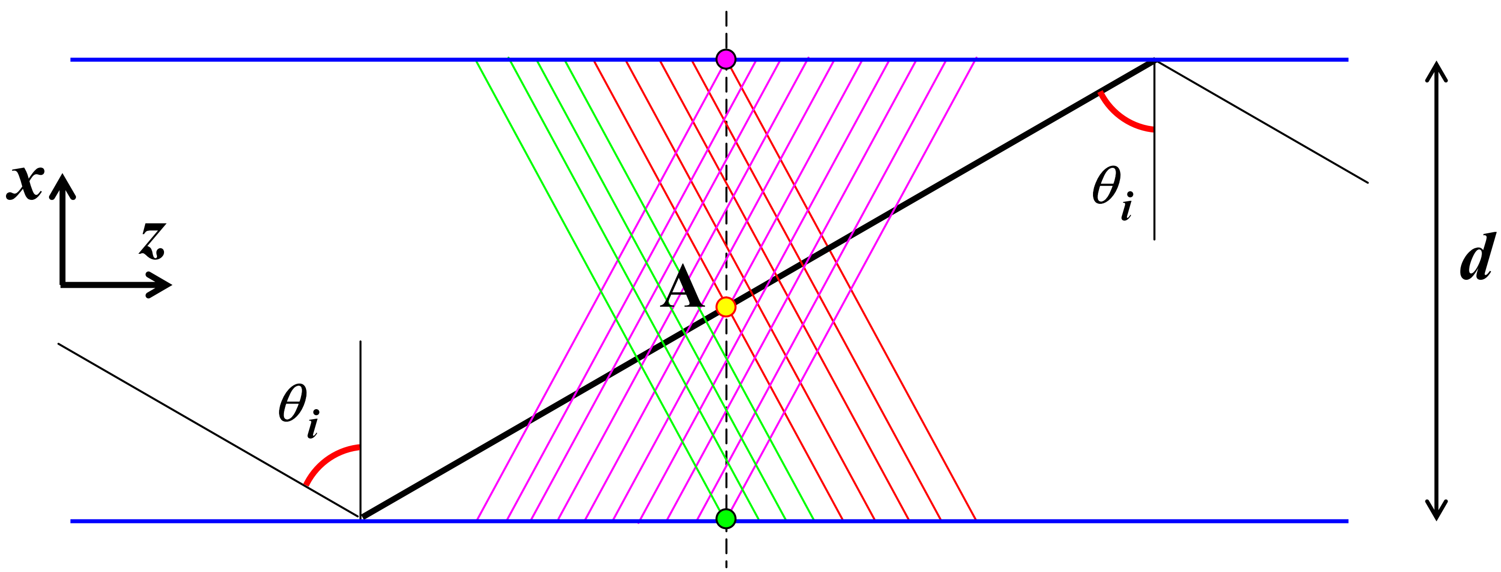
Similarly to the parallel plate waveguide, we assume propagation along the z–direction and uniform conditions along the y–direction.

Guided waves are launched with angle of incidence larger than the critical angle, so that total (internal) reflection takes place. Waves launched at smaller angles suffer partial refraction into the cladding and eventually the power in the core region will disappear for sufficiently long wave guides.



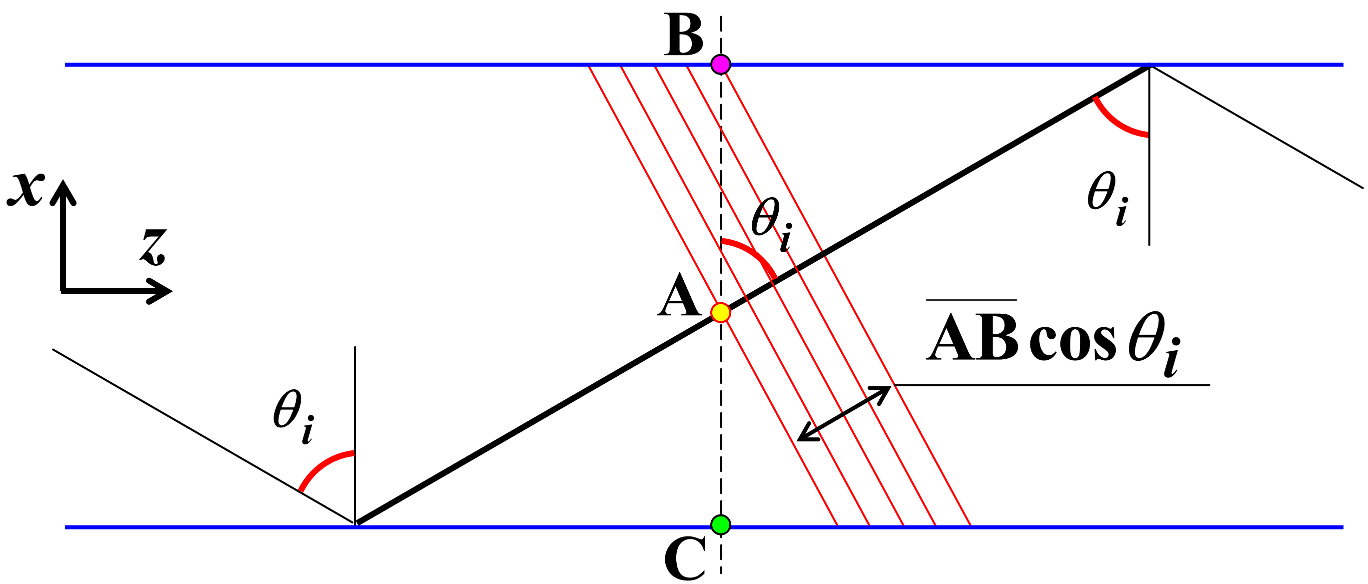
We consider again TE and TM modes. Only certain angles of incidence are allowed, but here the reflection coefficient for total reflection is a complex quantity, introducing a phase shift in the reflected field, which depends on the angle of incidence. In the case of metal plates, instead, there is always a phase shift of 180° for the tangential electric field.

In order for the angle to be accepted, the wave needs to establish a self–consistent constructive interference pattern for any point inside the core, as indicated in the figure below



Consider a point A in the core of the wave guide and a wave front moving from it reaching point B. The phase shift for the phase planes moving from A to B is

$$\Delta\varphi_1 = -\beta_1 \cdot \overline{\mathbf{AB}} \cos\theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\varepsilon_{r1}}} \overline{\mathbf{AB}} \cos\theta_i$$



λ_o is the wavelength in vacuum at the given frequency of operation.

The wave front reflected at point B experiences a phase jump equal to the phase of the complex reflection coefficient. Assuming a TE wave, or perpendicular polarization,

$$\Delta\varphi_2 = \angle\,\Gamma_\perp(E)_B = \angle\left(\frac{\sqrt{\varepsilon_1}\cos\theta_i + j\sqrt{\varepsilon_2}\sqrt{\varepsilon_1/\varepsilon_2\sin^2\theta_i - 1}}{\sqrt{\varepsilon_1}\cos\theta_i - j\sqrt{\varepsilon_2}\sqrt{\varepsilon_1/\varepsilon_2\sin^2\theta_i - 1}}\right)$$

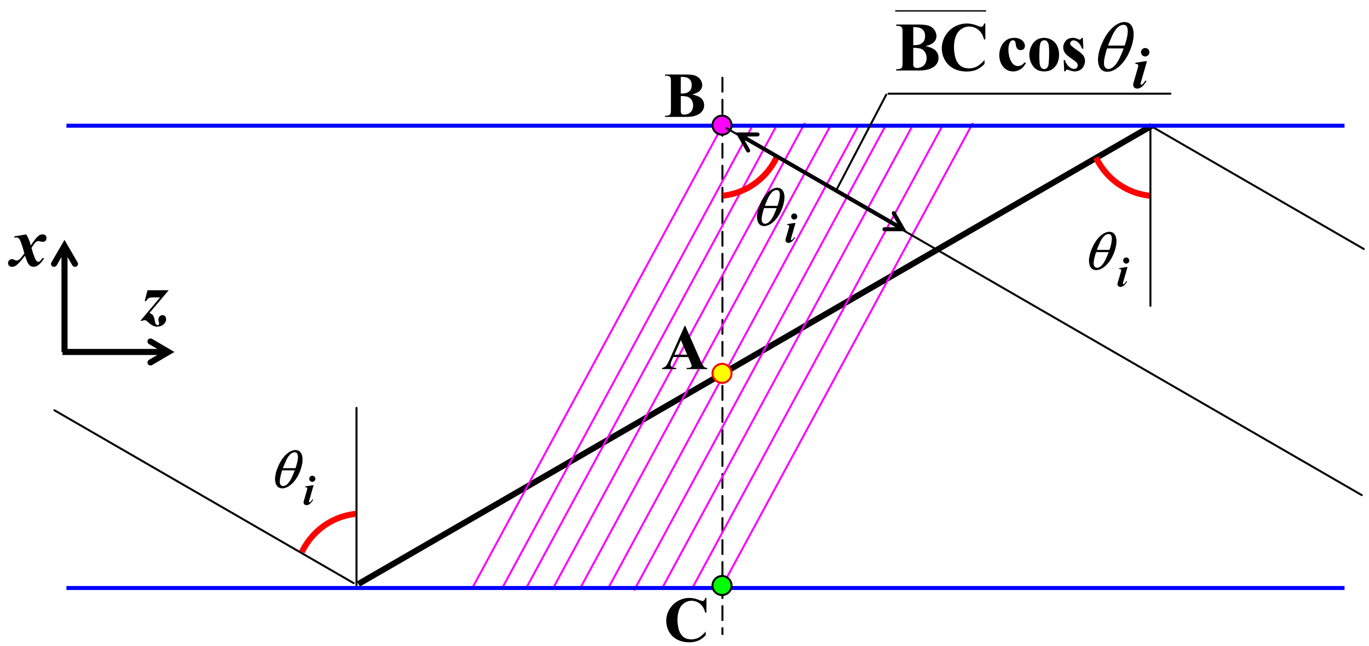
$$= \angle\left(\frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_1/\varepsilon_2}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_1/\varepsilon_2}}\right)$$

$$= 2\angle\left(\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_1/\varepsilon_2}\right)$$

$$= 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - \varepsilon_1/\varepsilon_2}}{\cos\theta_i}$$

Then, the reflected wave experiences a phase shift when moving from B to C

$$\Delta\varphi_3 = -\beta_1 \cdot \overline{BC}\cos\theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\varepsilon_{r1}}}\overline{BC}\cos\theta_i$$



The wave front reflected at point C experiences again a phase jump equal to the phase of the complex reflection coefficient. For a symmetric waveguide

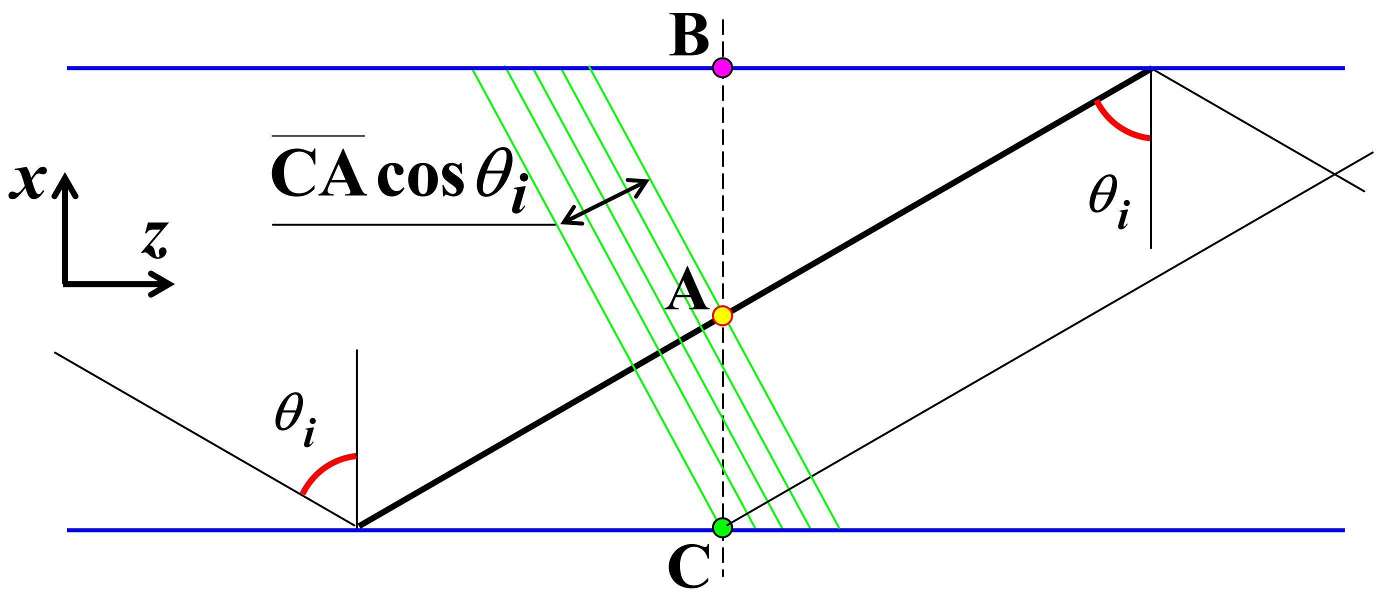
$$\Rightarrow \quad \Delta\varphi_4 = \Delta\varphi_2$$

$$\Delta\varphi_4 = \angle\,\Gamma_\perp(E)_C = \angle\left(\frac{\sqrt{\varepsilon_1}\cos\theta_i + j\sqrt{\varepsilon_2}\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}}{\sqrt{\varepsilon_1}\cos\theta_i - j\sqrt{\varepsilon_2}\sqrt{\dfrac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}}\right)$$

$$= 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i}$$

The reflected wave experiences a phase shift moving from C back to A

$$\Delta\varphi_5 = -\beta_1 \cdot \overline{\mathbf{CA}}\cos\theta_i = -\frac{2\pi}{\lambda_o / \sqrt{\varepsilon_{r1}}}\overline{\mathbf{CA}}\cos\theta_i$$



For constructive interference (self–consistency), the sum of all the phase shift components must be equal to a multiple of 2π

$$-\frac{2\pi}{\lambda_o/\sqrt{\varepsilon_{r1}}}\left(\overline{AB}+\overline{BC}+\overline{CA}\right)\cos\theta_i+\Delta\varphi_2+\Delta\varphi_4=-2m\pi,$$

$$m = 0, 1, 2\ldots$$

with $\left(\overline{AB}+\overline{BC}+\overline{CA}\right)=2d$

$$\Rightarrow \quad \boxed{\frac{2\pi d}{\lambda_o/\sqrt{\varepsilon_{r1}}}\cos\theta_i - m\pi = 2\tan^{-1}\frac{\sqrt{\sin^2\theta_i-\dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i} \qquad m = 0, 1, 2\ldots}$$

Taking the tangent of all terms we obtain the characteristic equation for the TE modes.

$$\tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i} \quad , \quad m = 0, 1, 2\ldots$$

In terms of even and odd solutions, we can rewrite

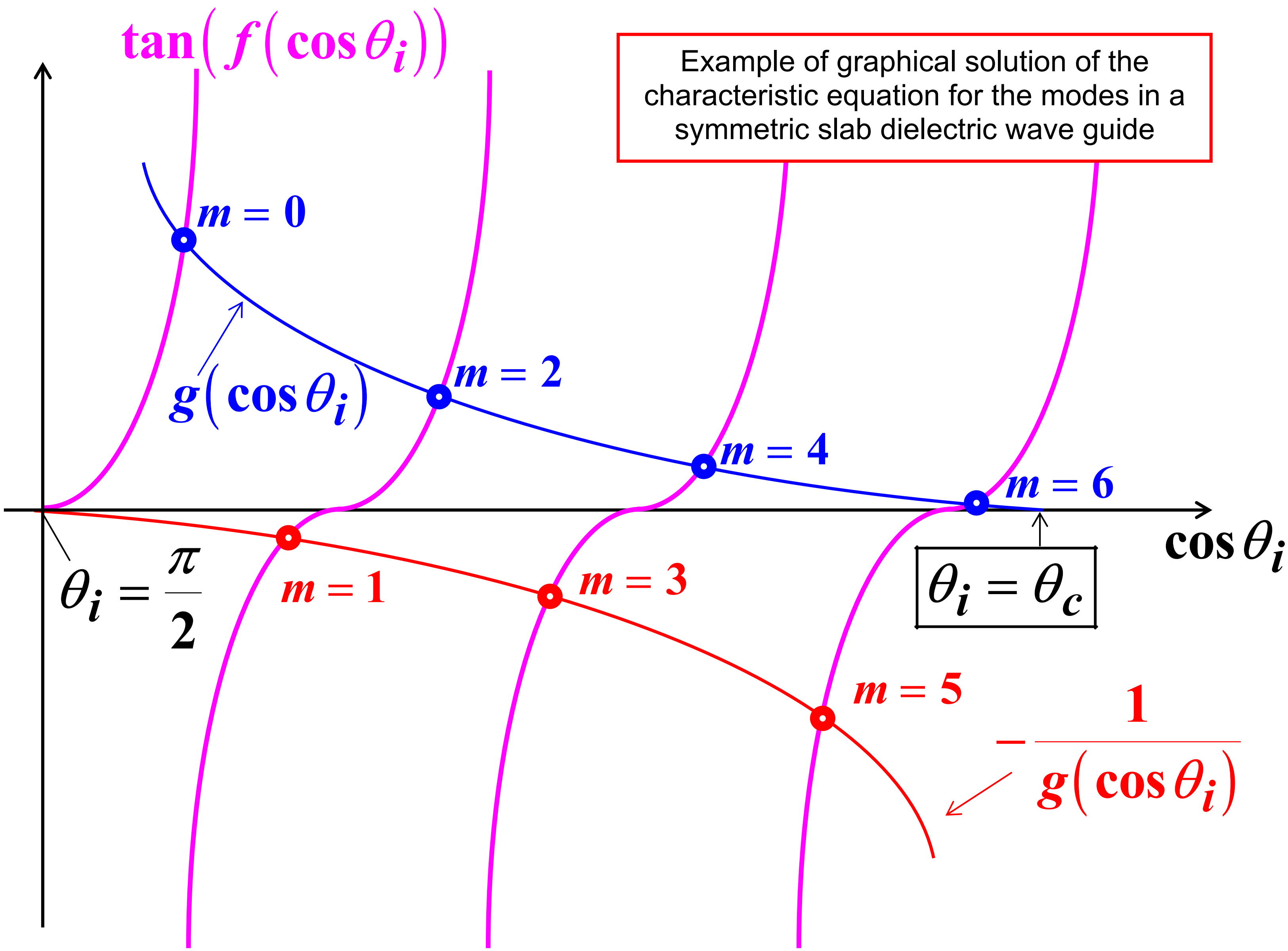
$$\tan\left(\underbrace{\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i}_{f(\cos\theta_i)}\right) = \begin{cases} \dfrac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_i} = g(\cos\theta_i) & \boxed{\text{Even modes}} \quad m = 0, 2, \ldots \\[2ex] -\dfrac{\cos\theta_i}{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}} = -\dfrac{1}{g(\cos\theta_i)} & \boxed{\text{Odd modes}} \quad m = 1, 3, \ldots \end{cases}$$

The characteristic equation for TM modes is obtained by using the reflection coefficient for parallel polarization in the derivation

$$\tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\left(\varepsilon_2/\varepsilon_1\right)\cos\theta_i} \quad , \quad m = 0, 1, 2\ldots$$

or, in terms of even and odd solutions

$$\tan\left(\underbrace{\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_o}\cos\theta_i}_{f(\cos\theta_i)}\right) = \begin{cases} \dfrac{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}}{\left(\varepsilon_2/\varepsilon_1\right)\cos\theta_i} = g(\cos\theta_i) & \text{Even modes} \quad m = 0, 2, \ldots \\ -\dfrac{\left(\varepsilon_2/\varepsilon_1\right)\cos\theta_i}{\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}} = -\dfrac{1}{g(\cos\theta_i)} & \text{Odd modes} \quad m = 1, 3, \ldots \end{cases}$$



$\tan\left(f\left(\cos\theta_i\right)\right)$
Example of graphical solution of the characteristic equation for the modes in a symmetric slab dielectric wave guide
$m = 0$
$m = 2$
$m = 4$
$m = 6$
$g\left(\cos\theta_i\right)$
$\cos\theta_i$
$\theta_i = \frac{\pi}{2}$
$m = 1$
$m = 3$
$m = 5$
$\theta_i = \theta_c$
$-\frac{1}{g\left(\cos\theta_i\right)}$

The cut-off frequencies for the modes are obtained by observing that at cut-off the angle of incidence is minimum (critical angle). At the critical angle, the characteristic equation is

$$\text{TE)}\quad \tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_{oc}}\cos\theta_c - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_c - \frac{\varepsilon_2}{\varepsilon_1}}}{\cos\theta_c} = 0$$

$$\text{TM)}\quad \tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_{oc}}\cos\theta_c - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_c - \frac{\varepsilon_2}{\varepsilon_1}}}{\underbrace{(\varepsilon_2/\varepsilon_1)\cos\theta_c}} = 0$$

since $\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$

for both TE and TM modes $\Rightarrow \quad \frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_{oc}}\cos\theta_c = \frac{m\pi}{2}$

The cut-off wavelengths (referenced to free space as usual in optical wave guides) and the corresponding cut-off frequencies for the guided modes are

$$\lambda_{oc} = \frac{2d\sqrt{\varepsilon_{r1}}}{m}\cos\theta_c = \frac{2d\sqrt{\varepsilon_{r1}}}{m}\sqrt{1-\sin^2\theta_c}$$

$$= \frac{2d\sqrt{\varepsilon_{r1}}}{m}\sqrt{1-\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \frac{2d}{m}\sqrt{\varepsilon_{r1}-\varepsilon_{r2}}$$

$$\boxed{f = \frac{c}{\lambda_{oc}} = \frac{mc}{2d\sqrt{\varepsilon_{r1}-\varepsilon_{r2}}}} \quad , \quad \boxed{m = 0, 1, 2\ldots}$$

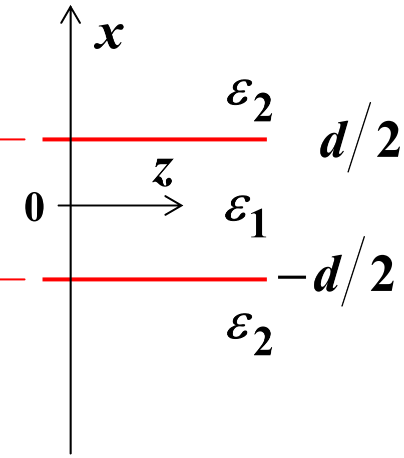
The fundamental modes are the TE_0 and the TM_0 with <u>zero</u> cut-off frequency.

TE and TM modes with the same index form <u>degenerate pairs</u> with identical cut-off frequencies.

Field expressions

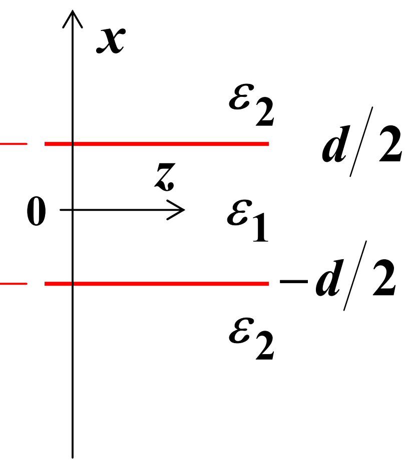
Even TE modes

$$E_y = \begin{cases} E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ E_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ E_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



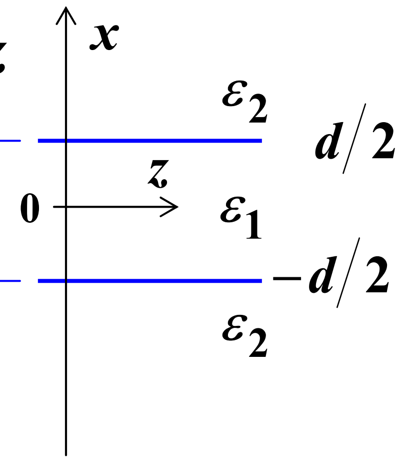
Odd TE modes

$$E_y = \begin{cases} E_o \sin\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ E_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ E_o \sin\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



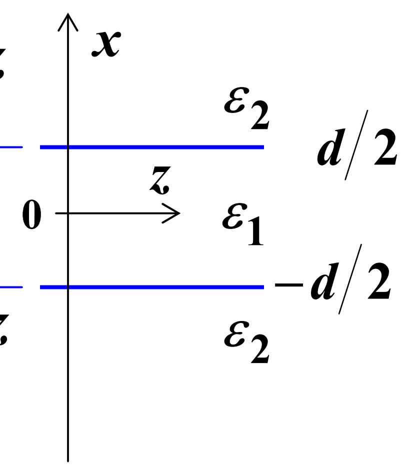
Even TM modes

$$H_y = \begin{cases} H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ H_o \cos\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ H_o \cos\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



Odd TM modes

$$H_y = \begin{cases} H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ H_o \sin\left(\beta_{x1} \cdot x\right) e^{-j\beta_z z} \\ -H_o \sin\left(\beta_{x1} \cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$



In medium 1

$$\left(\beta_1\right)^2 = \beta_{x1}^2 + \beta_z^2 = \omega\sqrt{\mu_o\,\varepsilon_{r1}\,\varepsilon_o}$$

In medium 2

$$\left(\beta_2\right)^2 = -\alpha_{x2}^2 + \beta_z^2 = \omega\sqrt{\mu_o\,\varepsilon_{r2}\,\varepsilon_o}$$

We have

$$\beta_z = \beta_1 \sin\theta_i$$

$$\beta_{x1} = \beta_1 \cos\theta_i$$

For each mode the angle of incidence is obtained from the solution of the characteristic equation.

Examples of profiles for the transverse electric field of TE modes. TM modes have similar profiles for the magnetic field.

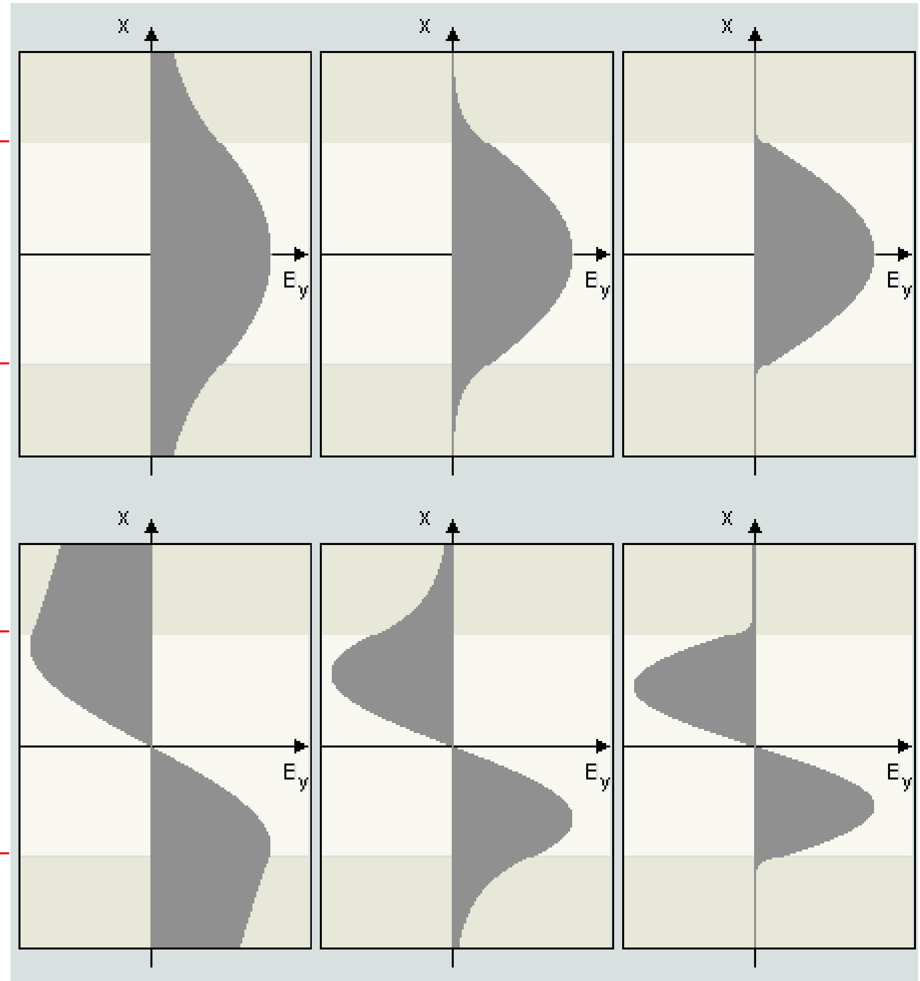
f just above cut-off → $f >> f_c$

↑ attenuation

↓ attenuation

↑ attenuation

↓ attenuation



TE_0

TE_1

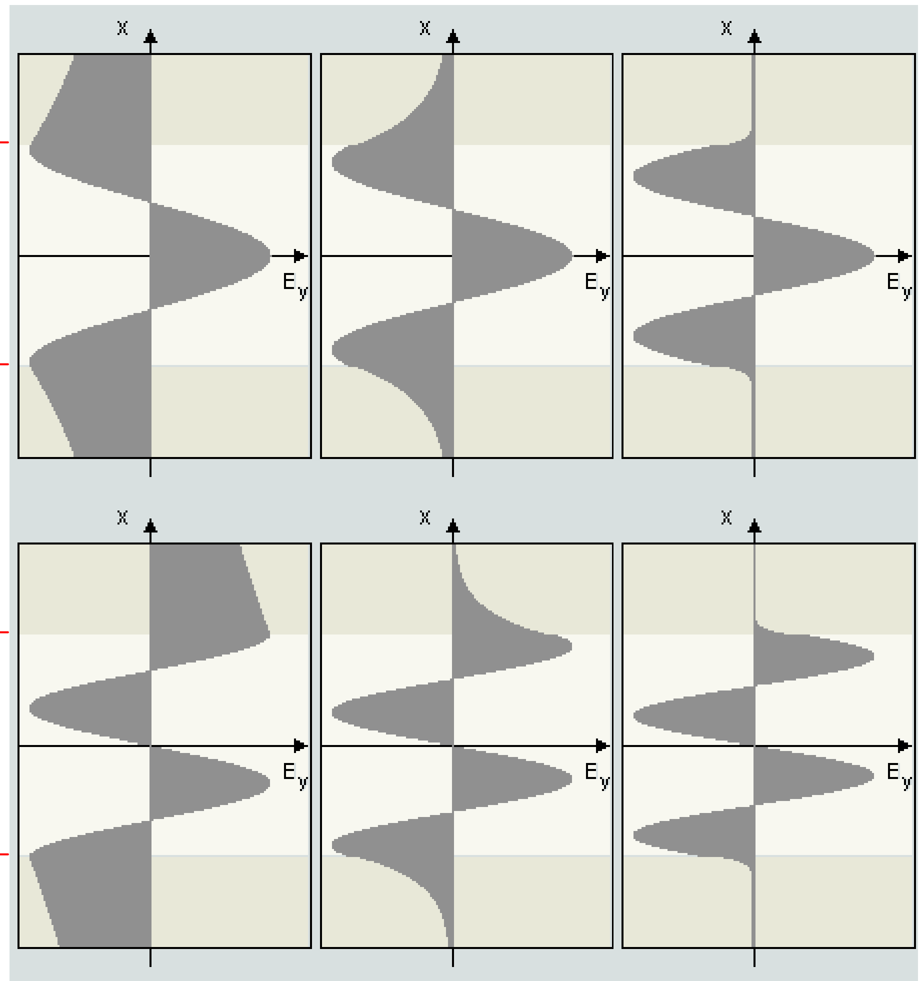
f **just above cut-off** → $f \gg f_c$

↑ **attenuation**

↓ **attenuation**

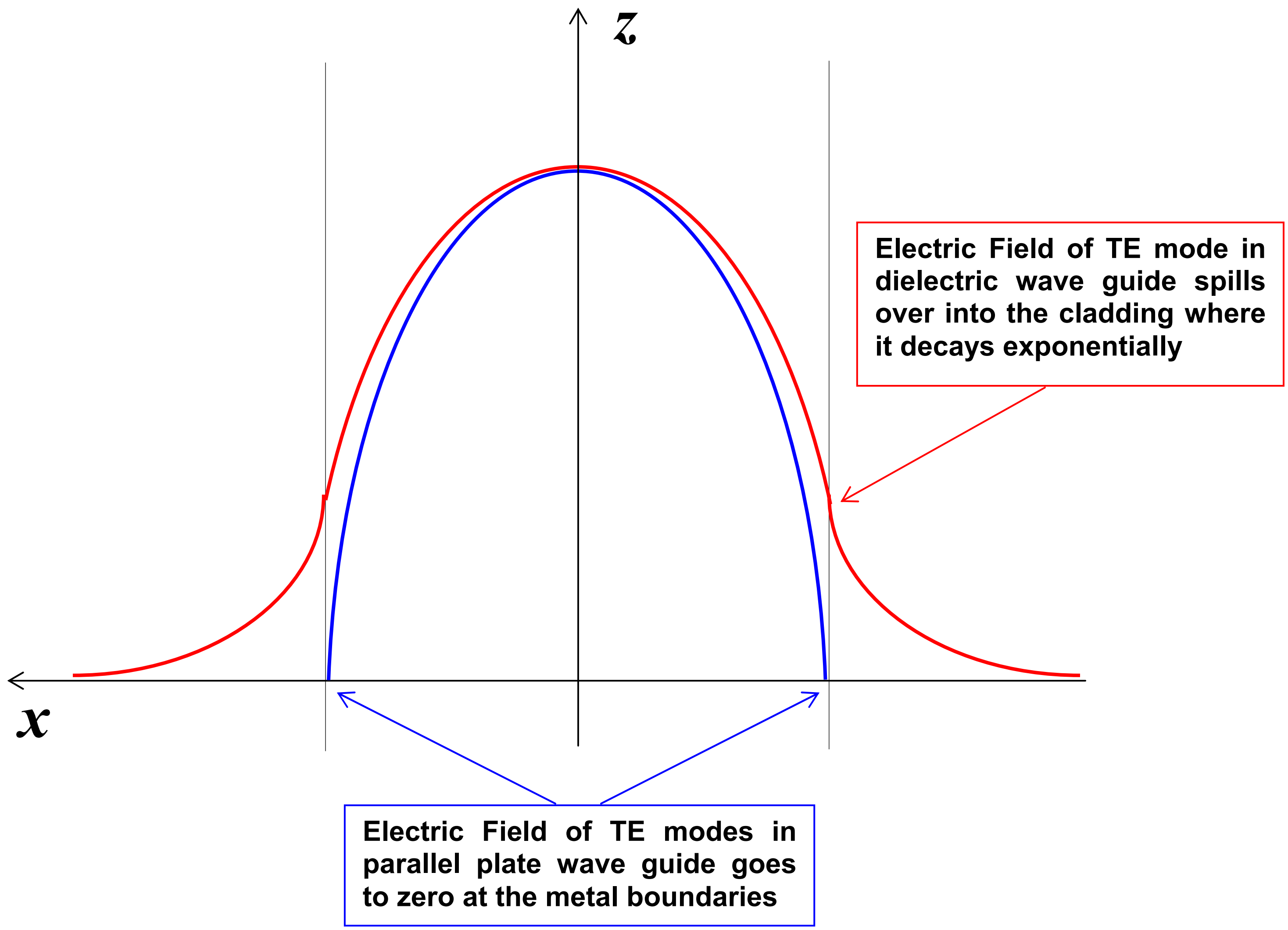
↑ **attenuation**

↓ **attenuation**



TE_2

TE_3


z
x
Electric Field of TE mode in dielectric wave guide spills over into the cladding where it decays exponentially
Electric Field of TE modes in parallel plate wave guide goes to zero at the metal boundaries

Magnetic field components for TE modes are obtained from Faraday's law

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\vec{\mathbf{H}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \cancel{\dfrac{\partial}{\partial y}} & \dfrac{\partial}{\partial z} \\ \mathbf{E}_x = 0 & \mathbf{E}_y & \mathbf{E}_z = 0 \end{bmatrix} \Rightarrow \begin{array}{l} -\dfrac{\partial}{\partial z}\mathbf{E}_y = -j\omega\mu_o\mathbf{H}_x \\ \dfrac{\partial}{\partial z}\mathbf{E}_x - \dfrac{\partial}{\partial x}\mathbf{E}_z = -j\omega\mu_o\mathbf{H}_y = 0 \\ \dfrac{\partial}{\partial x}\mathbf{E}_y = -j\omega\mu_o\mathbf{H}_z \end{array}$$

For example, the transverse magnetic field component is proportional to the (transverse) electric field. In the guide core:

$$\boxed{-\frac{\partial}{\partial z}\mathrm{E}_y = -j\omega\,\mu_o \mathrm{H}_x}$$

$$\left.\begin{array}{l} \textbf{Even)}\; -\dfrac{\partial}{\partial z} E_o \cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\[2ex] \textbf{Odd)}\; -\dfrac{\partial}{\partial z} E_o \sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \end{array}\right. = -j\omega\,\mu_o \mathrm{H}_x$$

$$\Rightarrow \quad \mathrm{H}_x = -\frac{\beta_z}{\omega\,\mu_o}\mathrm{E}_y = \begin{array}{ll} -\dfrac{\beta_z E_o}{\omega\,\mu_o}\cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} & \textbf{(Even)} \\[2ex] -\dfrac{\beta_z E_o}{\omega\,\mu_o}\sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} & \textbf{(Odd)} \end{array}$$

Electric field components for TM modes are obtained from Ampere's law

$$\nabla \times \vec{\mathbf{H}} = j\omega\varepsilon\vec{\mathbf{E}}$$

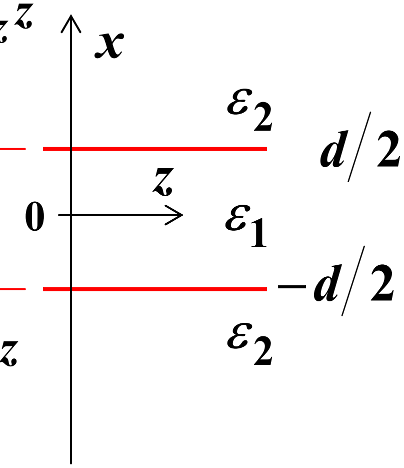
$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \cancel{\dfrac{\partial}{\partial y}} & \dfrac{\partial}{\partial z} \\ \mathbf{H}_x{=}0 & \mathbf{H}_y & \mathbf{H}_z{=}0 \end{bmatrix} \Rightarrow \begin{array}{c} -\dfrac{\partial}{\partial z}\mathbf{H}_y = j\omega\varepsilon\mathbf{E}_x \\ \dfrac{\partial}{\partial z}\mathbf{H}_x - \dfrac{\partial}{\partial x}\mathbf{H}_z = 0 = j\omega\varepsilon\mathbf{E}_y \\ \dfrac{\partial}{\partial x}\mathbf{H}_y - = j\omega\varepsilon\mathbf{E}_z \end{array}$$

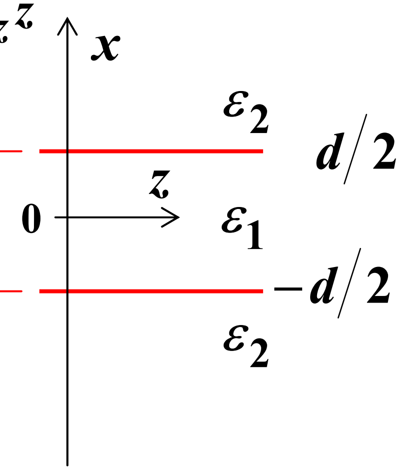
Next, is a summary of all the field components parallel to the plane of incidence.

Even TE modes

$$H_x = \begin{cases} -\dfrac{\beta_z}{\omega\mu_o} E_o \cos\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z / \omega\mu_o\right) E_o \cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ -\dfrac{\beta_z}{\omega\mu_o} E_o \cos\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$

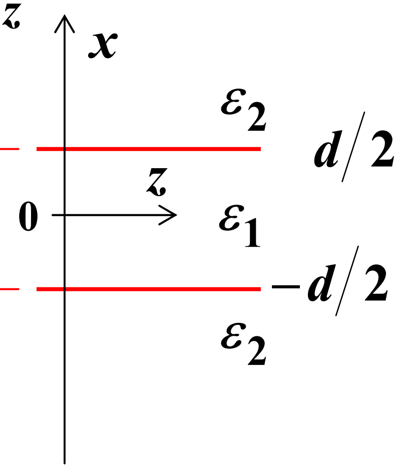


$$H_z = \begin{cases} -\dfrac{j\alpha_{x2}}{\omega\mu_o} E_o \cos\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -j\left(\beta_{x1} / \omega\mu_o\right) E_o \sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ \dfrac{j\alpha_{x2}}{\omega\mu_o} E_o \cos\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$

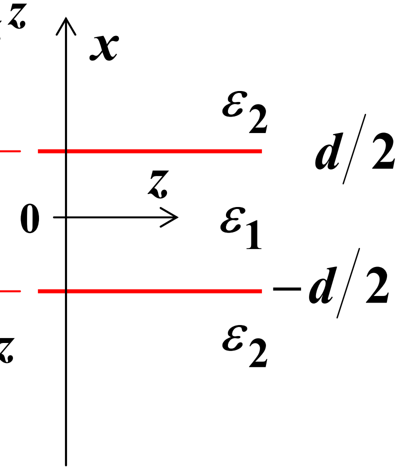


Odd TE modes

$$H_x = \begin{cases} -\dfrac{\beta_z}{\omega\,\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z \big/ \omega\,\mu_o\right) E_o \sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ \dfrac{\beta_z}{\omega\,\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$

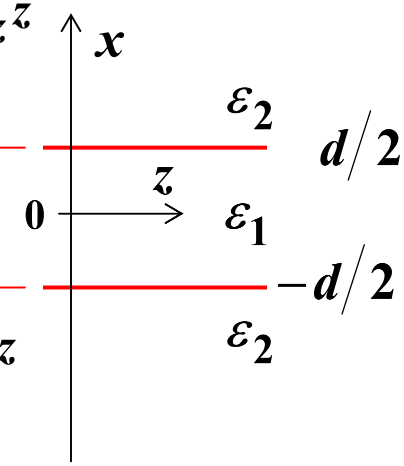


$$H_z = \begin{cases} -\dfrac{j\alpha_{x2}}{\omega\,\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ j\left(\beta_{x1} \big/ \omega\,\mu_o\right) E_o \cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ -\dfrac{j\alpha_{x2}}{\omega\,\mu_o} E_o \sin\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$

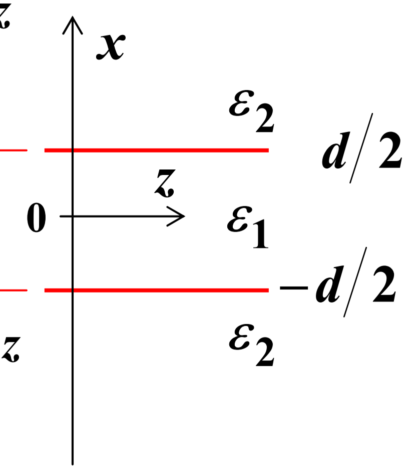


Even TM modes

$$E_x = \begin{cases} -\dfrac{\beta_z}{\omega\,\varepsilon_2} H_o \cos\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ -\left(\beta_z \big/ \omega\,\varepsilon_1\right) H_o \cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ -\dfrac{\beta_z}{\omega\,\varepsilon_2} H_o \cos\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$

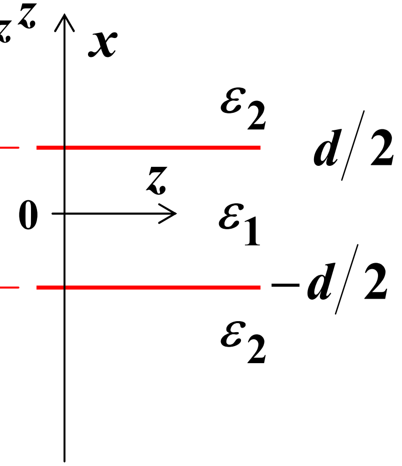


$$E_z = \begin{cases} \dfrac{j\alpha_{x2}}{\omega\,\varepsilon_2} H_o \cos\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)} e^{-j\beta_z z} \\ j\left(\beta_{x1} \big/ \omega\,\varepsilon_1\right) H_o \sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ -\dfrac{j\alpha_{x2}}{\omega\,\varepsilon_2} H_o \cos\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)} e^{-j\beta_z z} \end{cases}$$

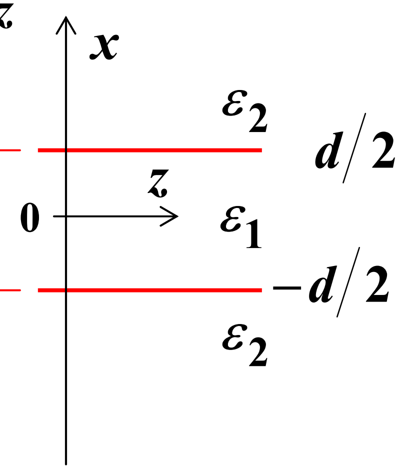


Odd TM modes

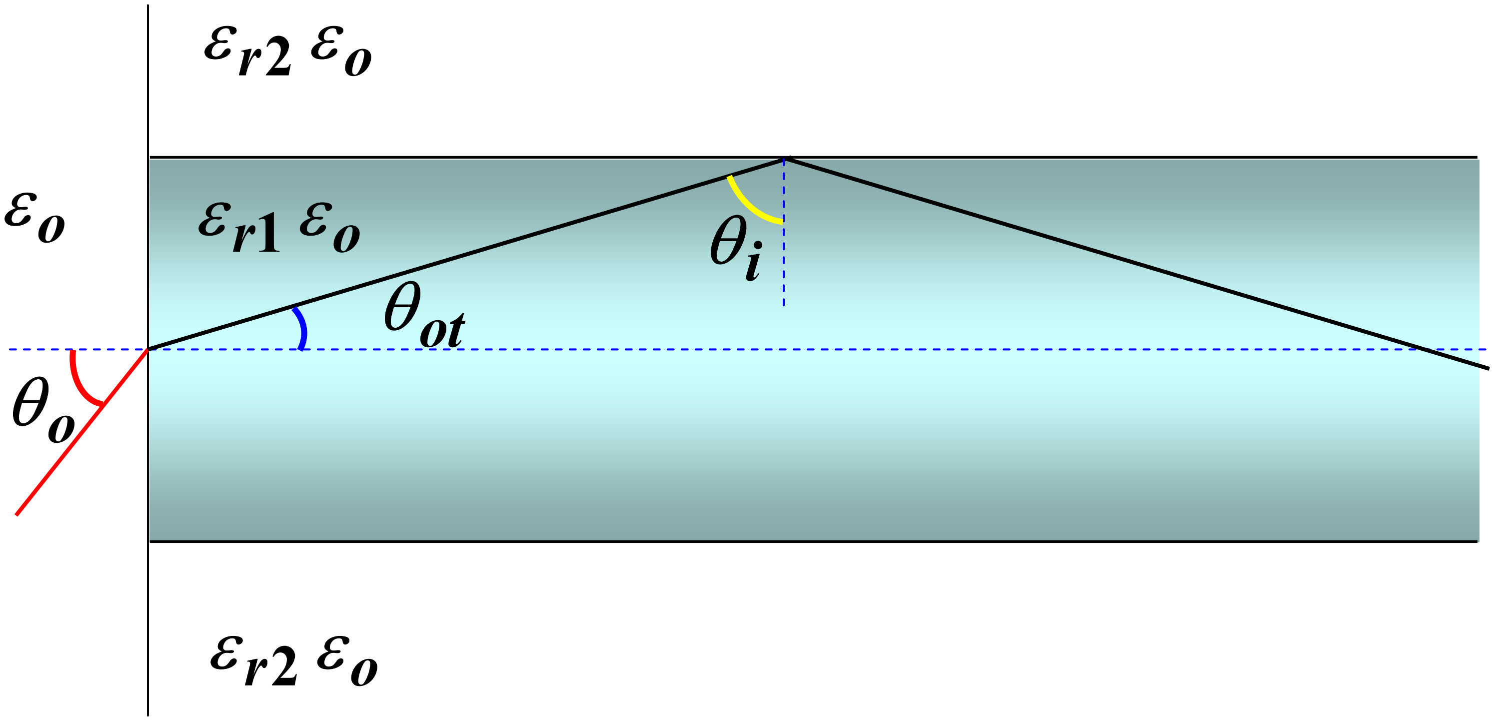
$$E_x = \begin{cases} -\dfrac{\beta_z}{\omega\,\varepsilon_2} H_o \sin\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)}\, e^{-j\beta_z z} \\ -\left(\beta_z \big/ \omega\,\varepsilon_1\right) H_o \sin\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ \dfrac{\beta_z}{\omega\,\varepsilon_2} H_o \sin\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)}\, e^{-j\beta_z z} \end{cases}$$



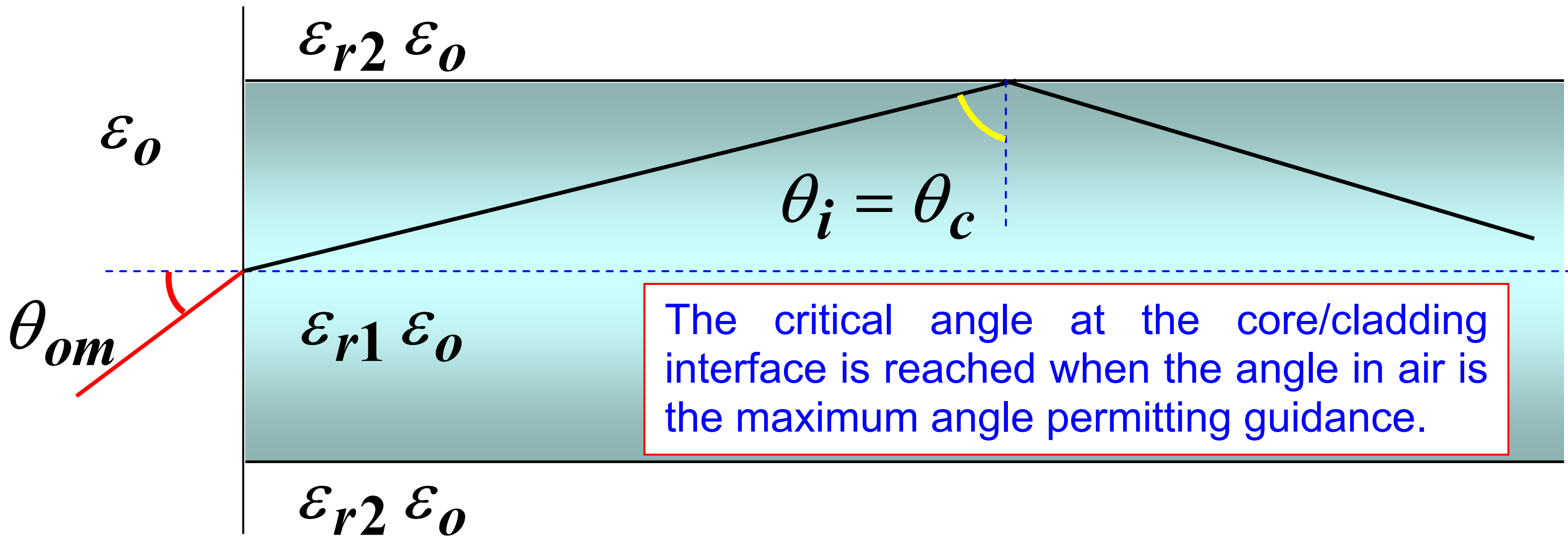
$$E_z = \begin{cases} \dfrac{j\alpha_{x2}}{\omega\,\varepsilon_2} H_o \sin\left(\beta_{x1}\cdot d/2\right) e^{-\alpha_{x2}(x-d/2)}\, e^{-j\beta_z z} \\ -j\left(\beta_{x1} \big/ \omega\,\varepsilon_1\right) H_o \cos\left(\beta_{x1}\cdot x\right) e^{-j\beta_z z} \\ \dfrac{j\alpha_{x2}}{\omega\,\varepsilon_2} H_o \sin\left(\beta_{x1}\cdot d/2\right) e^{\alpha_{x2}(x+d/2)}\, e^{-j\beta_z z} \end{cases}$$

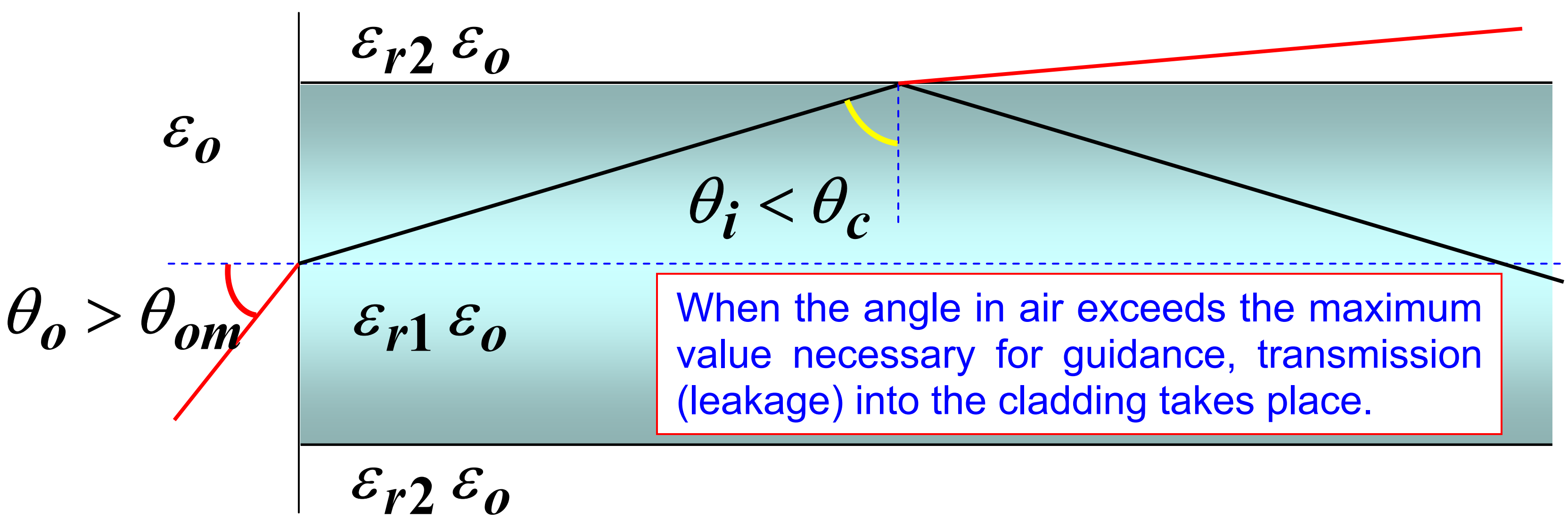


Consider a wave entering the end of the wave guide from air.



A classical problem of optical waveguides is to determine the maximum angle of entrance θ_o that satisfies the condition of total internal reflection (guidance).

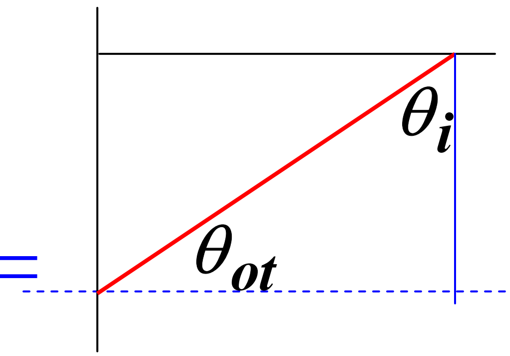

$\varepsilon_{r2}\,\varepsilon_o$
ε_o
$\theta_i = \theta_c$
θ_{om}
$\varepsilon_{r1}\,\varepsilon_o$
The critical angle at the core/cladding interface is reached when the angle in air is the maximum angle permitting guidance.
$\varepsilon_{r2}\,\varepsilon_o$


$\varepsilon_{r2}\,\varepsilon_o$
ε_o
$\theta_i < \theta_c$
$\theta_o > \theta_{om}$
$\varepsilon_{r1}\,\varepsilon_o$
When the angle in air exceeds the maximum value necessary for guidance, transmission (leakage) into the cladding takes place.
$\varepsilon_{r2}\,\varepsilon_o$

At the air–core interface

$$\sin\theta_{ot} = \sqrt{\frac{\varepsilon_{rair}\varepsilon_o}{\varepsilon_{r1}\varepsilon_o}}\sin\theta_o = \sqrt{\frac{1}{\varepsilon_{r1}}}\sin\theta_o$$

$$\theta_i + \theta_{ot} = 90° \quad \Rightarrow \quad \cos\theta_{ot} = \sin\theta_i \quad \Leftarrow$$



At the critical angle

$$\sin\theta_i = \sin\theta_c = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

$$\sin^2\theta_{otm} = 1 - \cos^2\theta_{otm} = 1 - \sin^2\theta_c = 1 - \frac{\varepsilon_{r2}}{\varepsilon_{r1}} = \frac{\sin^2\theta_{om}}{\varepsilon_{r1}}$$

$$\Rightarrow \quad \sin\theta_{om} = \sqrt{\varepsilon_{r1} - \varepsilon_{r2}} = \textbf{numerical aperture}$$

$$\theta_{om} = \sin^{-1}\sqrt{\varepsilon_{r1} - \varepsilon_{r2}}$$