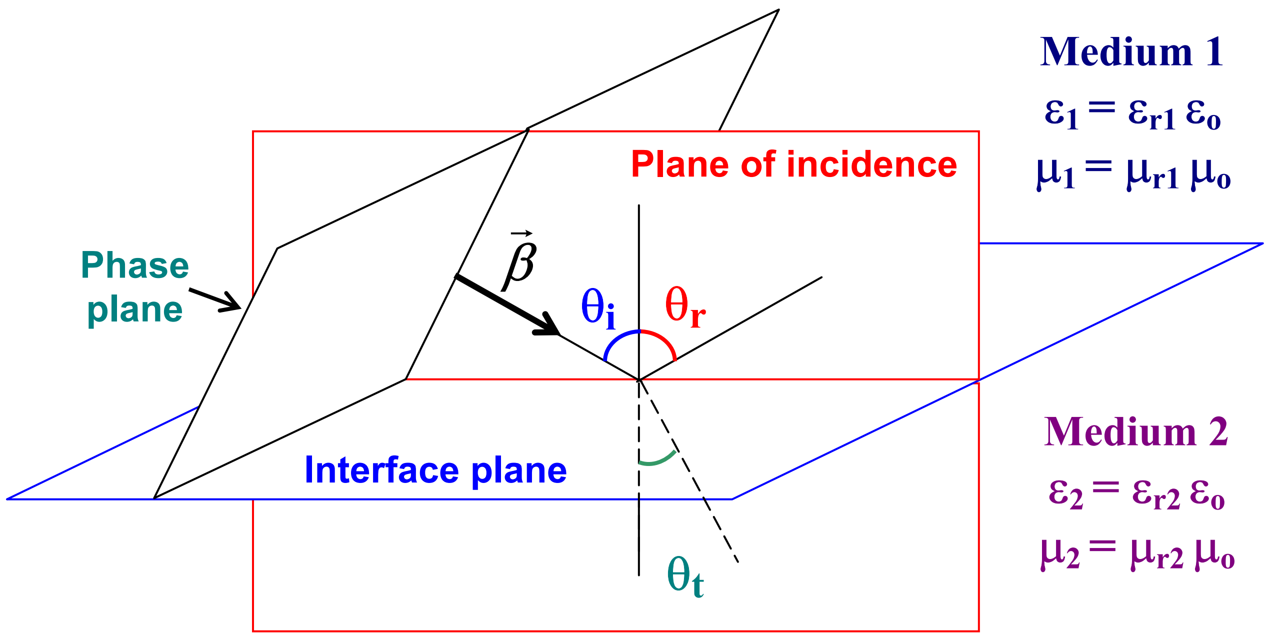
Oblique incidence: Interface between dielectric media

Consider a planar interface between two dielectric media. A plane wave is incident at an angle from medium 1.

- **The interface plane defines the boundary between the media.**
- **The plane of incidence contains the propagation vector and is both perpendicular to the interface plane and to the phase planes of the wave.**



There are two elementary orientations (polarizations) for the electromagnetic fields:

- **Perpendicular Polarization**

  The electric field is perpendicular to the plane of incidence and the magnetic field is parallel to the plane of incidence.

  The fields are configured as in the Transverse Electric (TE) modes.

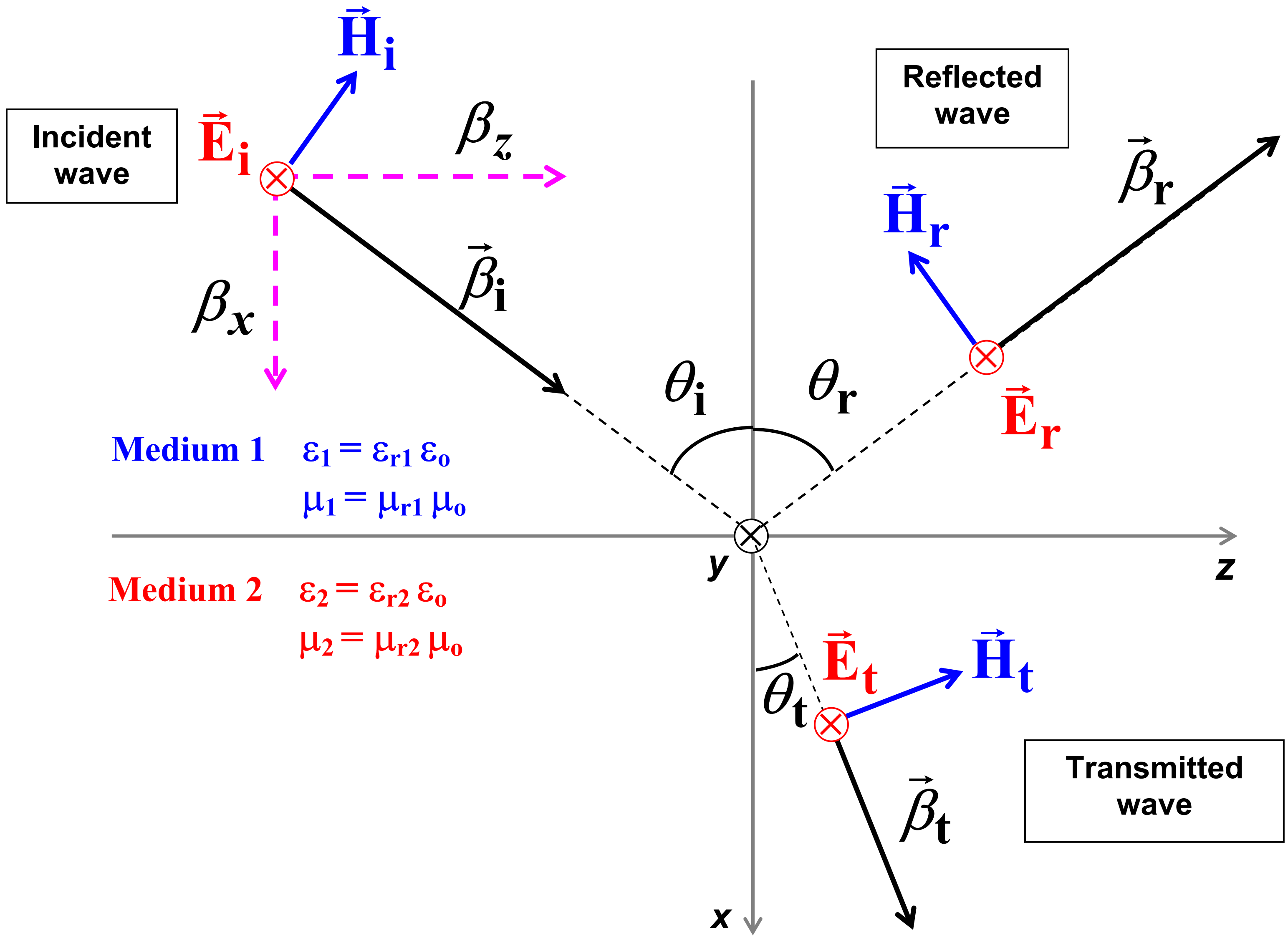
- **Parallel Polarization**

  The magnetic field is perpendicular to the plane of incidence and the electric field is parallel to the plane of incidence.

  The fields are configured as in the Transverse Magnetic (TM) modes.

Any plane wave with general field orientation can be obtained by superposition of two waves with perpendicular and parallel polarization.

Perpendicular (TE) polarization



The electric field phasors for the perpendicular polarization, with reference to the system of coordinates in the figure, are given by

$$\vec{\mathrm{E}}_{\mathrm{i}} = E_{yi}\, e^{-j\beta_{ix}\cdot x - j\beta_{iz}\cdot z}\ \hat{i}_y$$

$$\vec{\mathrm{E}}_{\mathrm{r}} = E_{yr}\, e^{-j\beta_{rx}\cdot x - j\beta_{rz}\cdot z}\ \hat{i}_y$$

$$\vec{\mathrm{E}}_{\mathrm{t}} = E_{yt}\, e^{-j\beta_{tx}\cdot x - j\beta_{tz}\cdot z}\ \hat{i}_y$$

The propagation vector components in medium 1 are expressed as

$$\left|\vec{\beta}_i\right| = \sqrt{\beta_{ix}^2 + \beta_{iz}^2} = \beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

$$\beta_{ix} = \beta_1 \cos\theta_i \qquad \beta_{iz} = \beta_1 \sin\theta_i$$

$$\left|\vec{\beta}_r\right| = \sqrt{\beta_{rx}^2 + \beta_{rz}^2} = \beta_1$$

$$\beta_{rx} = -\beta_1 \cos\theta_r \qquad \beta_{rz} = \beta_1 \sin\theta_r$$

The propagation vector components in medium 2 are expressed as

$$\left|\vec{\beta}_t\right| = \sqrt{\beta_{tx}^2 + \beta_{tz}^2} = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

$$\beta_{tx} = \beta_2 \cos\theta_t \qquad \beta_{tz} = \beta_2 \sin\theta_t$$

The magnetic field components can be obtained as

$$\vec{\mathrm{H}}_{\mathrm{i}} = \frac{\vec{\beta}_i \times \vec{\mathrm{E}}_{\mathrm{i}}}{\omega\mu_1} = \frac{E_{yi}}{\eta_1}\left(-\sin\theta_i\ \hat{i}_x + \cos\theta_i\ \hat{i}_z\right) e^{-j\beta_{ix}x - j\beta_{iz}z}$$

$$\vec{\mathrm{H}}_{\mathrm{r}} = \frac{\vec{\beta}_r \times \vec{\mathrm{E}}_{\mathrm{r}}}{\omega\mu_1} = -\frac{E_{yr}}{\eta_1}\left(\sin\theta_r\ \hat{i}_x + \cos\theta_r\ \hat{i}_z\right) e^{-j\beta_{rx}x - j\beta_{rz}z}$$

$$\vec{\mathrm{H}}_{\mathrm{t}} = \frac{\vec{\beta}_t \times \vec{\mathrm{E}}_{\mathrm{t}}}{\omega\mu_2} = \frac{E_{yt}}{\eta_2}\left(-\sin\theta_t\ \hat{i}_x + \cos\theta_t\ \hat{i}_z\right) e^{-j\beta_{tx}x - j\beta_{tz}z}$$

Assuming that the amplitude of the incident electric field is given, to completely specify the problem we need to find the amplitude of reflected and transmitted electric field.

The boundary condition at the interface (*x* = 0) states that the tangential electric field must be continuous. Because of the perpendicular polarization, the tangential field is also the total field

$$x = 0) \qquad E_{yi}\, e^{-j\beta_{iz} z} + E_{yr}\, e^{-j\beta_{rz} z} = E_{yt}\, e^{-j\beta_{tz} z}$$

The relation above must be valid for any choice of "*z*" and we must have (phase conservation law)

$$\beta_{iz} = \beta_{rz} = \beta_{tz}$$

The first equality indicates that the reflected angle is the same as the incident angle.

$$\beta_{iz} = \beta_{rz} \Rightarrow \quad \beta_1 \sin\theta_i = \beta_1 \sin\theta_r \quad \Rightarrow \quad \theta_i = \theta_r$$

The second equality provides the transmitted angle

$$\beta_{iz} = \beta_{tz} \quad \Rightarrow \quad \beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$

$$\Rightarrow \quad \theta_t = \sin^{-1}\left(\sqrt{\frac{\mu_1\,\varepsilon_1}{\mu_2\,\varepsilon_2}}\sin\theta_i\right)$$

Snell's Law

Since we have also

$$e^{-j\beta_{iz}z} = e^{-j\beta_{rz}z} = e^{-j\beta_{tz}z}$$

the boundary condition for the electric field becomes

$$E_{yi} + E_{yr} = E_{yt}$$

The tangential magnetic field must also be continuous at the interface. This applies in our case to the *z*–components

$$H_{zi} \quad + \quad H_{zr} \qquad = H_{zt}$$

$$\frac{E_{yi}}{\eta_1}\cos\theta_i - \frac{E_{yr}}{\eta_1}\cos\theta_i = \frac{E_{yt}}{\eta_2}\cos\theta_t$$

$$\Rightarrow \quad E_{yi} - E_{yr} = \frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} E_{yt}$$

Solution of the system of boundary equations gives

$$\Gamma_\perp(E) = \frac{E_{yr}}{E_{yi}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$ **Reflection coefficient**

$$\tau_\perp(E) = \frac{E_{yt}}{E_{yi}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$ **Transmission coefficient**

For the magnetic field, we can define the reflection coefficient as

$$\Gamma_\perp(H) = \frac{H_{zr}}{H_{zi}} = -\frac{H_r}{H_i}$$

In terms of electric field, the magnetic field components are

$$H_{zr} = \frac{-E_{yr}}{\eta_1}\cos\theta_i = -H_r\cos\theta_i$$

$$H_{zi} = \frac{E_{yi}}{\eta_1}\cos\theta_i = H_i\cos\theta_i$$

The reflection coefficient for the magnetic field is then

$$\Gamma_\perp(H) = \frac{-E_{yr}}{E_{yi}} = -\Gamma_\perp(E) = \frac{\eta_1\cos\theta_t - \eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

The transmission coefficient is defined as

$$\tau_{\perp}(H) = \frac{H_t}{H_i}$$

The magnetic field components are

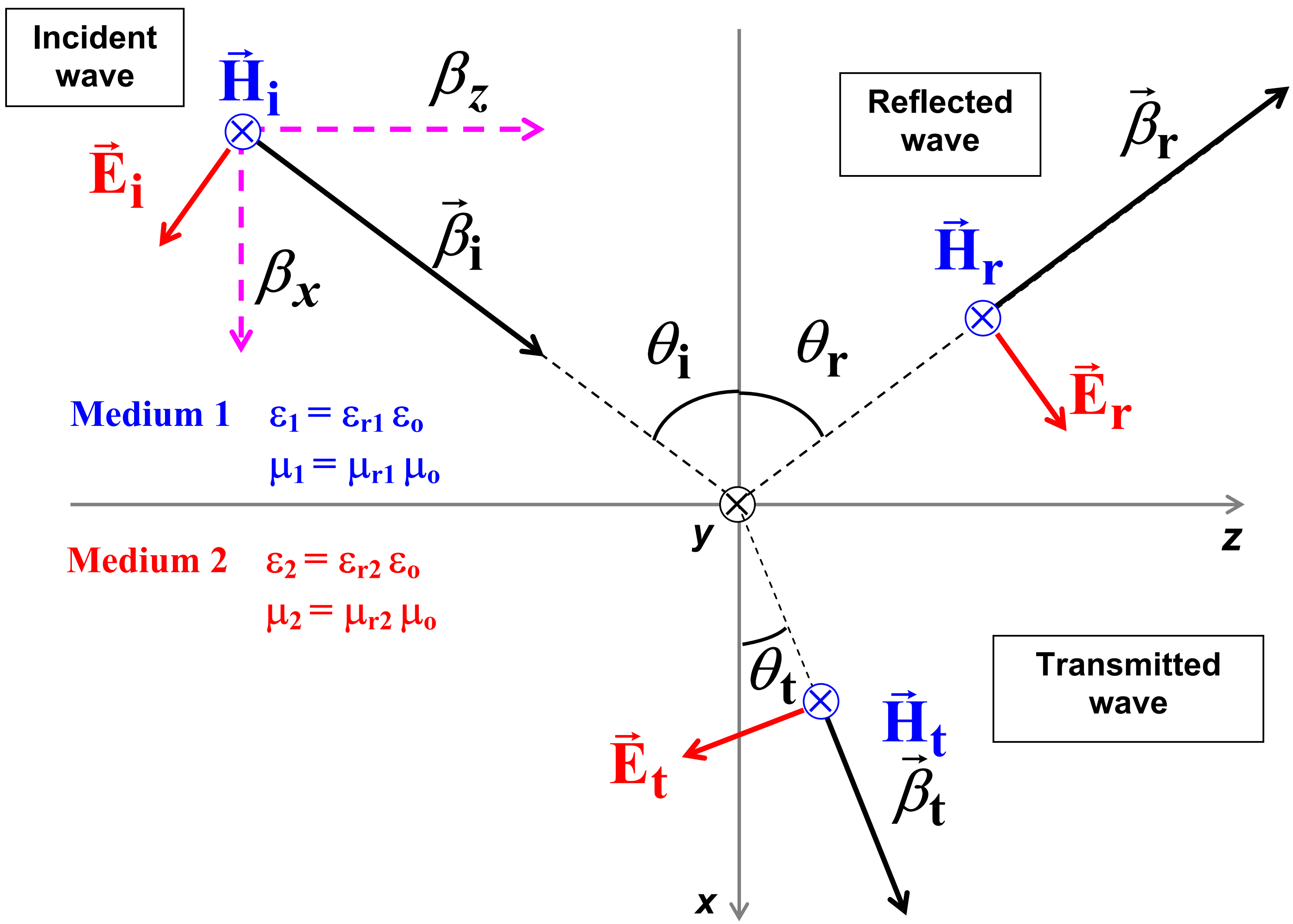
$$H_t = \frac{E_{yt}}{\eta_2}$$

$$H_i = \frac{E_{yi}}{\eta_1}$$

The transmission coefficient for the magnetic field is then

$$\tau_{\perp}(H) = \frac{H_t}{H_i} = \frac{\eta_1}{\eta_2}\tau_{\perp}(E) = \frac{2\eta_1 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

Parallel (TM) polarization



The magnetic field phasors for the parallel polarization are given by

$$\vec{\mathrm{H}}_{\mathrm{i}} = H_{yi}\, e^{-j\beta_{ix}\cdot x - j\beta_{iz}\cdot z}\ \hat{i}_y$$

$$\vec{\mathrm{H}}_{\mathrm{r}} = H_{yr}\, e^{-j\beta_{rx}\cdot x - j\beta_{rz}\cdot z}\ \hat{i}_y$$

$$\vec{\mathrm{H}}_{\mathrm{t}} = H_{yt}\, e^{-j\beta_{tx}\cdot x - j\beta_{tz}\cdot z}\ \hat{i}_y$$

and the electric field components can be obtained as

$$\vec{\mathrm{E}}_{\mathrm{i}} = -\frac{\vec{\beta}_i \times \vec{\mathrm{H}}_{\mathrm{i}}}{\omega\varepsilon_1} = \eta_1 H_{yi}\left(\sin\theta_i\ \hat{i}_x - \cos\theta_i\ \hat{i}_z\right) e^{-j\beta_{ix}x - j\beta_{iz}z}$$

$$\vec{\mathrm{E}}_{\mathrm{r}} = -\frac{\vec{\beta}_r \times \vec{\mathrm{H}}_{\mathrm{r}}}{\omega\varepsilon_1} = \eta_1 H_{yr}\left(\sin\theta_r\ \hat{i}_x + \cos\theta_r\ \hat{i}_z\right) e^{-j\beta_{rx}x - j\beta_{rz}z}$$

$$\vec{\mathrm{E}}_{\mathrm{t}} = -\frac{\vec{\beta}_t \times \vec{\mathrm{H}}_{\mathrm{t}}}{\omega\varepsilon_2} = \eta_2 H_{yt}\left(\sin\theta_t\ \hat{i}_x - \cos\theta_t\ \hat{i}_z\right) e^{-j\beta_{tx}x - j\beta_{tz}z}$$

Also for parallel polarization one can verify that the same relationships between angles apply, as found earlier for the perpendicular polarization, including Snell's law

$$\theta_i = \theta_r$$

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\mu_1\,\varepsilon_1}{\mu_2\,\varepsilon_2}}\sin\theta_i\right)$$

We have again two boundary conditions at the interface. One condition is for continuity of the tangential magnetic field

$$H_{yi} + H_{yr} = H_{yt}$$

A second condition is for continuity of the tangential electric field

$$E_{zi} \quad + \quad E_{zr} \quad = \quad E_{zt}$$

$$-\eta_1 \cos\theta_i H_{yi} + \eta_1 \cos\theta_i H_{yr} = -\eta_2 \cos\theta_t H_{yt}$$

$$\Rightarrow \quad H_{yi} - H_{yr} = \frac{\eta_2 \cos\theta_t}{\eta_1 \cos\theta_i} H_{yt}$$

From the equations provided by the boundary conditions we obtain the reflection and transmission coefficients for the magnetic field of a wave with parallel polarization as

$$\Gamma_{\|}(H) = \frac{H_{yr}}{H_{yi}} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

$$\tau_{\|}(H) = \frac{H_{yt}}{H_{yi}} = \frac{2\eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

The reflection coefficient for the electric field is defined as

$$\Gamma_{||}(E) = \frac{E_{zr}}{E_{zi}} = -\frac{E_r}{E_i}$$

The tangential components of the electric field can be expressed in terms of magnetic field as

$$E_{zr} = E_r \cos\theta_i = \eta_1 \cos\theta_i H_{yr}$$

$$E_{zi} = -E_i \cos\theta_i = -\eta_1 \cos\theta_i H_{yi}$$

The reflection coefficient for the electric field is

$$\Gamma_{||}(E) = \frac{E_{zr}}{E_{zi}} = -\frac{H_{yr}}{H_{yi}} = -\Gamma_{||}(H) = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

The transmission coefficient for the electric field is defined as

$$\tau_{||}(E) = \frac{E_t}{E_i}$$

The electric field components are given by

$$E_t = \eta_2 H_{yt}$$

$$E_i = \eta_1 H_{yi}$$

The transmission coefficient for the electric field becomes

$$\tau_{||}(E) = \frac{E_t}{E_i} = \frac{\eta_2 H_{yt}}{\eta_1 H_{yi}} = \frac{\eta_2}{\eta_1} \tau_{||}(H)$$

$$= \frac{\eta_2}{\eta_1} \frac{2\eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

Considerable simplifications are possible for the common case of nonmagnetic dielectric media with

$$\mu_1 = \mu_2 = \mu_o$$

First of all, Snell's law becomes

$$\theta_t = \sin^{-1}\left(\sqrt{\frac{\mu_1\,\varepsilon_1}{\mu_2\,\varepsilon_2}}\sin\theta_i\right) = \sin^{-1}\left(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i\right)$$

or, equivalently

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \qquad (n = \textbf{index of refraction})$$

Snell's law provides then a useful recipe to express the reflection and transmission coefficients only with angles, thus eliminating the explicit dependence on medium impedance.

After some trigonometric manipulations, we obtain the following table of simplified coefficients for electric and magnetic field

$$\Gamma_{\perp}(E) = -\Gamma_{\perp}(H) = -\frac{\sin\left(\theta_i - \theta_t\right)}{\sin\left(\theta_i + \theta_t\right)}$$

$$\Gamma_{||}(E) = -\Gamma_{||}(H) = -\frac{\tan\left(\theta_i - \theta_t\right)}{\tan\left(\theta_i + \theta_t\right)}$$

$$\tau_{\perp}(E) = \tau_{\perp}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\sin\theta_t\cos\theta_i}{\sin\left(\theta_i + \theta_t\right)}$$

$$\tau_{||}(E) = \tau_{||}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\sin\theta_t\cos\theta_i}{\sin\left(\theta_i + \theta_t\right)\cos\left(\theta_i - \theta_t\right)}$$

Power flow

The time-average power flow normal to the interface must be continuous. We can express this as

$$\frac{1}{2}\frac{E_i^2}{\eta_1}\cos\theta_i - \frac{1}{2}\frac{E_r^2}{\eta_1}\cos\theta_i = \frac{1}{2}\frac{E_t^2}{\eta_2}\cos\theta_t$$

incident power − reflected power = transmitted power

We define the reflection and transmission coefficients for the time-average power as

$$R = \frac{\text{reflected power}}{\text{incident power}} = \frac{E_r^2}{E_i^2}$$

$$T = \frac{\text{transmitted power}}{\text{incident power}} = \frac{E_t^2}{E_i^2}\frac{\eta_1}{\eta_2}\frac{\cos\theta_t}{\cos\theta_i}$$

The following conversion formulas relate power and electric field coefficients

$$R = \left| \Gamma^2(E) \right|$$

$$T = \left| \tau^2(E) \frac{\eta_1}{\eta_2} \frac{\cos\theta_t}{\cos\theta_i} \right|$$

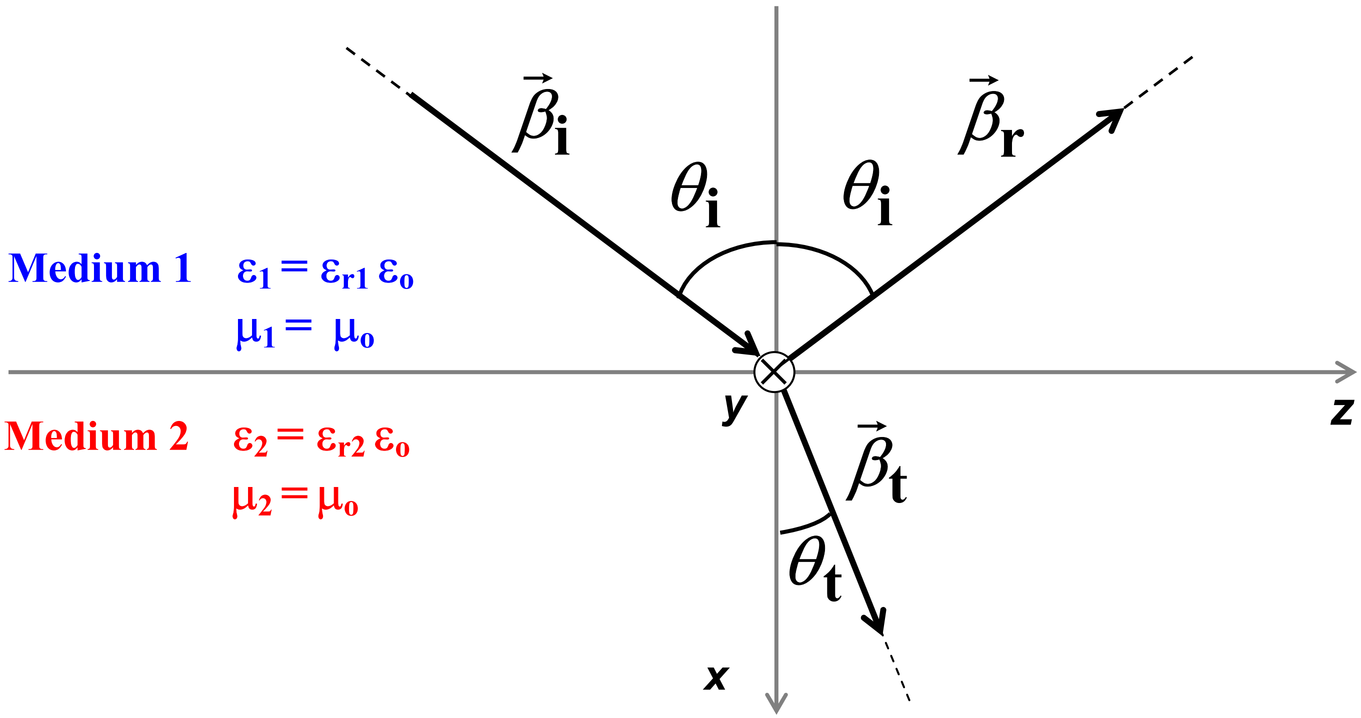
Note that reflection and transmission coefficients for the time-average power are always real positive quantities. The following power conservation condition is always verified

$$R + T = 1$$

Since the power flow normal to the interface is considered, the results obtained above apply equally to perpendicular and parallel polarization.

Non–magnetic perfect dielectric media

Case $\varepsilon_2 > \varepsilon_1$



From Snell's law

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t \qquad \varepsilon_2 > \varepsilon_1 \Rightarrow \theta_t < \theta_i$$

Since $\theta_t < \theta_i$ there is always a transmitted (refracted) beam.

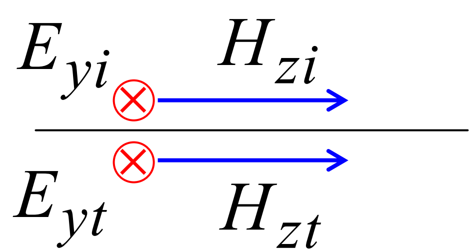
The transmission coefficients are always positive

$$\tau_{\perp}(E) = \tau_{\perp}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)} > 0$$

$$\tau_{||}(E) = \tau_{||}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} > 0$$

$\Rightarrow$ **transmitted and incident wave are in phase at the boundary.**

$$\tau_{\perp}(E) = \frac{E_{yt}}{E_{yi}}$$



Perpendicular polarization

$$\Gamma_{\perp}(E) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \qquad \Gamma_{\perp}(H) = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

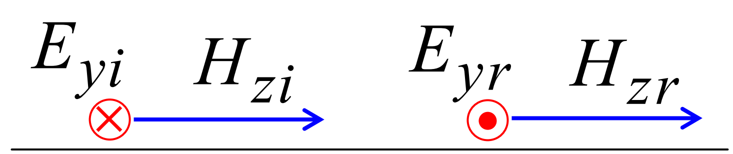
- **The reflection coefficient for the electric field is always negative**

E_{yi} and E_{yr} have always phase difference of 180°

- **The reflection coefficient for the magnetic field is always positive**

H_{zi} and H_{zr} are always in phase

$$\Gamma_{\perp}(E) = \frac{E_{yr}}{E_{yi}} \qquad \Gamma_{\perp}(H) = \frac{H_{zr}}{H_{zi}}$$



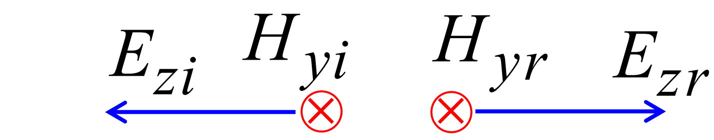
Parallel polarization

$$\Gamma_{\parallel}(E) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \qquad \Gamma_{\parallel}(H) = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

When $\theta_i + \theta_t < 90° \quad \Rightarrow \quad \tan(\theta_i + \theta_t) > 0$

E_{zi} and E_{zr} have phase difference of 180°

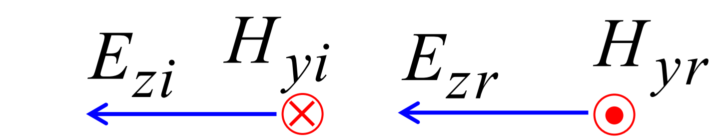
H_{yi} and H_{yr} are in phase



When $\theta_i + \theta_t > 90° \quad \Rightarrow \quad \tan(\theta_i + \theta_t) < 0$

E_{zi} and E_{zr} are in phase

H_{yi} and H_{yr} have phase difference of 180°



$$\text{When} \quad \theta_i + \theta_t = 90° \quad \Rightarrow \quad \tan(\theta_i + \theta_t) \rightarrow \infty$$

the reflection coefficients vanish (TOTAL TRANSMISSION)

$$\Gamma_{||}(E) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \rightarrow 0\,; \quad \Gamma_{||}(H) = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \rightarrow 0$$

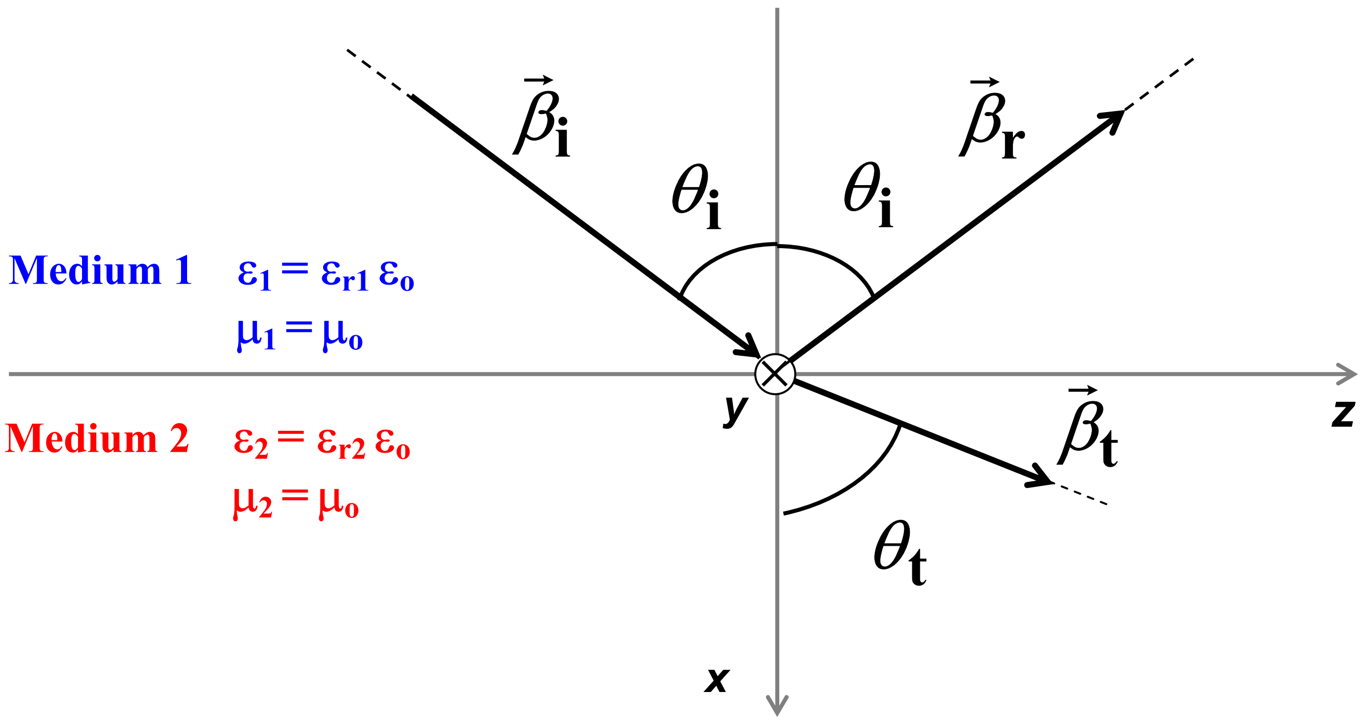
$$\text{For} \quad \theta_i + \theta_t = 90° \quad \Rightarrow \quad \theta_i = \theta_B \quad \text{(Brewster angle)}$$

$$\text{and} \ \Rightarrow \quad \cos\theta_B = \sin\theta_t$$

From Snell's law

$$\frac{\sin\theta_B}{\sin\theta_t} = \frac{\sin\theta_B}{\cos\theta_B} = \tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \quad \Rightarrow \quad \theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Case $\varepsilon_2 < \varepsilon_1$



From Snell's law

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t \qquad \varepsilon_2 < \varepsilon_1 \Rightarrow \theta_t > \theta_i$$

For angles of incidence such that

$$\sin\theta_t < 1$$

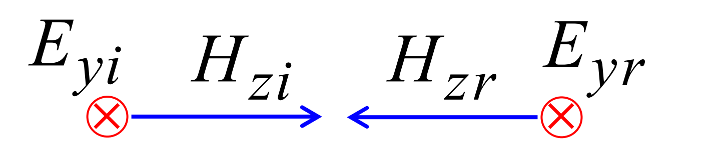
we have for perpendicular polarization

$$\Gamma_{\perp}(E) = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_i + \theta_t)} > 0$$

E_{yi} and E_{yr} are always in phase

$$\Gamma_{\perp}(H) = \frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = -\frac{\sin(\theta_t - \theta_i)}{\sin(\theta_i + \theta_t)} < 0$$

H_{zi} and H_{zr} have always phase difference of 180°



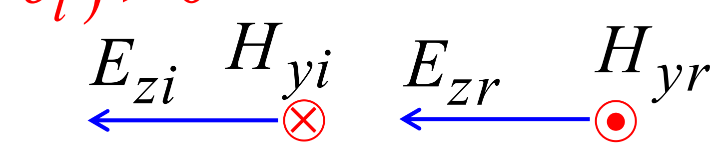
For parallel polarization

$$\Gamma_{\parallel}(E) = -\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)}$$

$$\Gamma_{\parallel}(H) = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = -\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)}$$

When $\theta_i + \theta_t < 90^\circ \quad \Rightarrow \quad \tan(\theta_i + \theta_t) > 0$

E_{zi} and E_{zr} are in phase

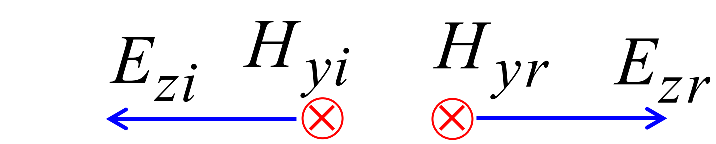


H_{yi} and H_{yr} have phase difference of 180°

When $\theta_i + \theta_t > 90^\circ \quad \Rightarrow \quad \tan(\theta_i + \theta_t) < 0$

E_{zi} and E_{zr} have phase difference of 180°

H_{yi} and H_{yr} are in phase



When $\theta_i + \theta_t = 90° \quad \Rightarrow \quad \tan(\theta_i + \theta_t) \rightarrow \infty$

Also in this case the reflection coefficients vanish and we have TOTAL TRANSMISSION

$$\Gamma_{||}(E) = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)} \rightarrow 0 \; ; \quad \Gamma_{||}(H) = -\frac{\tan(\theta_t - \theta_i)}{\tan(\theta_i + \theta_t)} \rightarrow 0$$

Total transmission occurs again, for parallel polarization only, at the Brewster angle

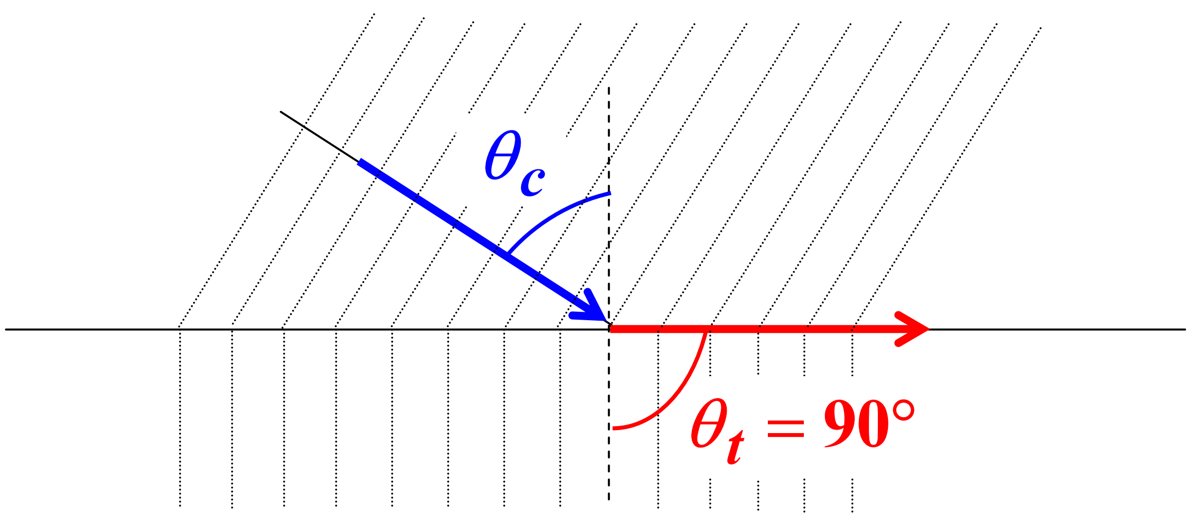
$$\theta_i = \theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

When

$$\sin\theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\sin\theta_t = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \Rightarrow \sin\theta_t = 1 \Rightarrow \theta_t = 90°$$

we have a limit condition for TOTAL REFLECTION, valid for both polarizations. This particular angle of incidence is called

critical angle $\theta_i = \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$



For angles of incidence beyond the critical angle

$$\theta_i > \theta_c \quad \Rightarrow \quad \sin\theta_t = \sin\theta_i \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} > 1$$

$$\Rightarrow \quad \cos\theta_t = \text{ imaginary}$$

$\pm$ **, choose "$-$"**

$$\cos\theta_t = -\sqrt{1-\sin^2\theta_t} = -j\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{\sin^2\theta_i - \frac{\varepsilon_2}{\varepsilon_1}}$$

The negative sign is selected, in order to obtain the proper wave vector in medium 2, as shown later.

The reflection and transmission coefficients become complex

$$\Gamma_\perp(E) = -\Gamma_\perp(H) = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

$$\Gamma_{||}(E) = -\Gamma_{||}(H) = \frac{-\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

$$\tau_\perp(E) = \tau_\perp(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\cos\theta_i}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

$$\tau_{||}(E) = \tau_{||}(H)\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}} = \frac{2\sqrt{\varepsilon_2/\varepsilon_1}\cos\theta_i}{\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}$$

If we consider the coefficients for time-average power flow, we have, for both polarizations

$$R = \Gamma(E) \cdot \Gamma^{*}(E) = 1$$

$$T = 1 - R = 0$$

This means that incident and reflected waves carry the same time-average power, and no power is transmitted to medium 2. But this does not mean that the field disappears in medium 2. The instantaneous power that enters medium 2 is eventually reflected back to medium 1.

The electric field phasor of the transmitted wave has the form

$$\mathrm{E_t} = E_t\, e^{-j\beta_{t\,x} \cdot x} e^{-j\beta_{t\,z} \cdot z} = E_t\, e^{-j\beta_2 \cos\theta_t \cdot x} e^{-j\beta_2 \sin\theta_t \cdot z}$$

The wave vector components are

$$\beta_2 \cos\theta_t = \omega\sqrt{\mu_o\varepsilon_2}\left(-j\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sqrt{\sin^2\theta_i - \frac{\varepsilon_2}{\varepsilon_1}}\right)$$

$$= -j\omega\sqrt{\mu_o\varepsilon_1}\sqrt{\sin^2\theta_i - \frac{\varepsilon_2}{\varepsilon_1}} = -j\beta_1\sqrt{\sin^2\theta_i - \frac{\varepsilon_2}{\varepsilon_1}} = -j\alpha_t$$

$$\beta_2 \sin\theta_t = \omega\sqrt{\mu_o\varepsilon_2}\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i = \beta_1\sin\theta_i = \beta_{iz}$$

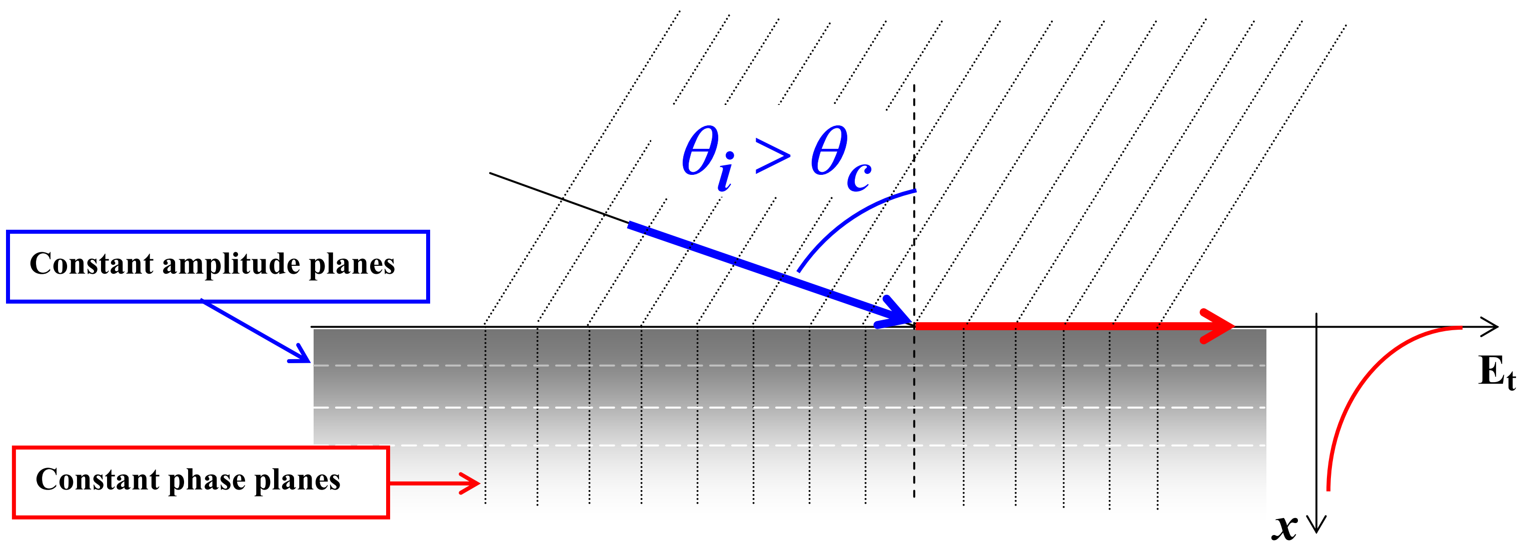
The field in medium 2 corresponds to a surface wave, moving along the z–direction and exponentially decaying (evanescent) along the x–direction

$$\mathrm{E}_t = E_t\, e^{-j(-j\alpha_t\cdot x)} e^{-j\beta_{iz}\cdot z} = E_t\, e^{-\alpha_t\cdot x} e^{-j\beta_{iz}\cdot z}$$

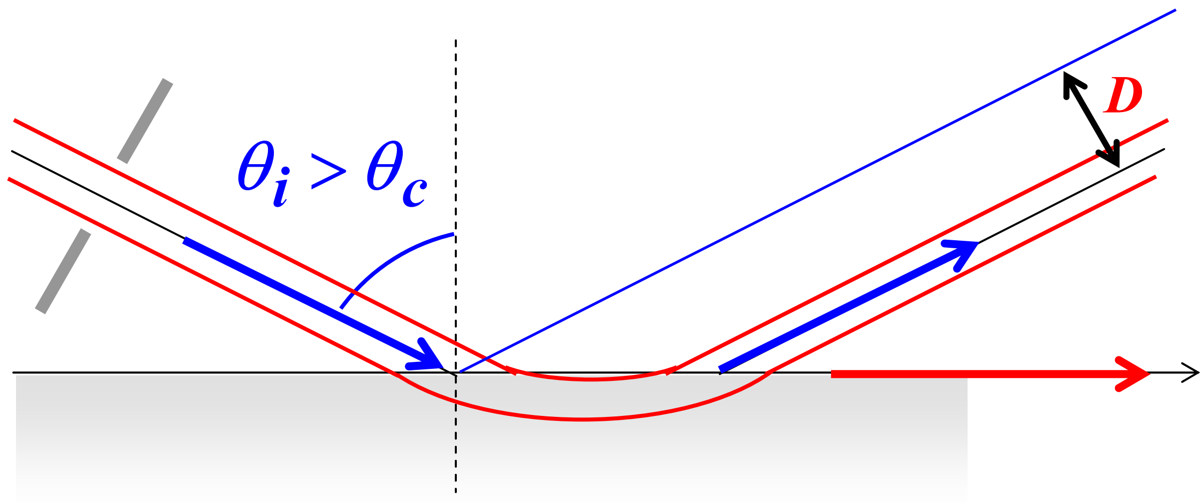
The surface wave moves parallel to the surface, with a phase velocity equal to the apparent phase velocity along z of the incident wave in medium 1

$$v_{p2} = \frac{v_{p1}}{\sin\theta_i} = v_{pz} > v_{p1}$$

For the surface wave, planes of constant amplitude are parallel and planes of constant phase are normal to the interface. These planes do not coincide, therefore the surface wave is a nontransverse wave.



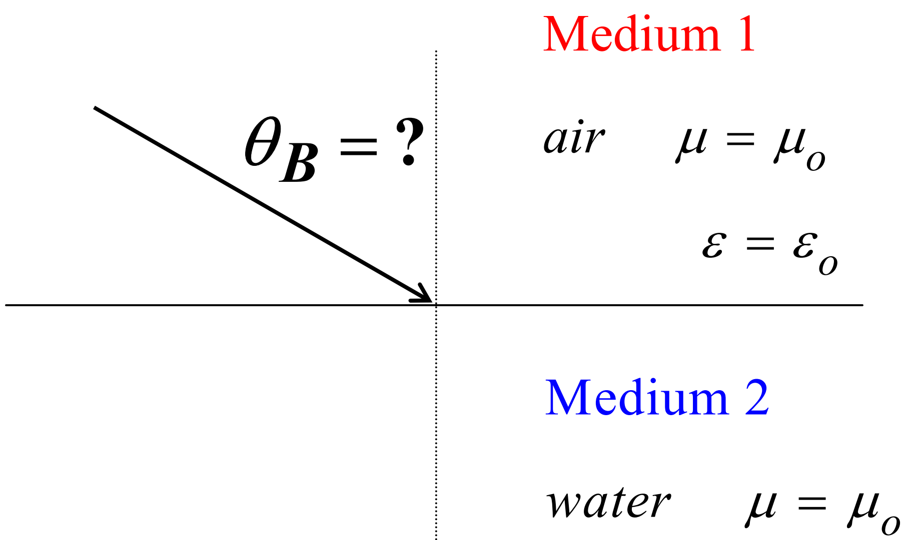
If you consider a beam incident on the interface, it is found that the power is totally reflected but after penetrating for some distance into medium 2. The reflected beam emerges displaced by a distance *D* (called Goos-Hänchen shift, discovered in 1947)



From experiments, the displacement is found to be

$$D \approx 0.52\sqrt{\varepsilon_2}\,\frac{2\pi}{\beta_1\sqrt{\sin^2\theta_i - \dfrac{\varepsilon_2}{\varepsilon_1}}} = 0.52\sqrt{\varepsilon_2}\,\frac{2\pi}{\alpha_t}$$

Examples:



$$\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\varepsilon = \begin{cases} 80\varepsilon_o & \text{microwaves} \\ 1.8\varepsilon_o & \text{optical} \end{cases}$$

$$\theta_B = \tan^{-1}\sqrt{80} \cong 83.6^\circ \quad \text{microwaves}$$

$$\theta_B = \tan^{-1}\sqrt{1.8} \cong 53.3^\circ \quad \text{optical}$$

At the Brewster angle

$$\theta_i + \theta_t = \theta_B + \theta_t = 90^\circ$$

$$\Rightarrow \theta_t \cong \begin{cases} 6.38^\circ & \text{microwaves} \\ 36.7^\circ & \text{optical} \end{cases}$$

Verification with Snell's law

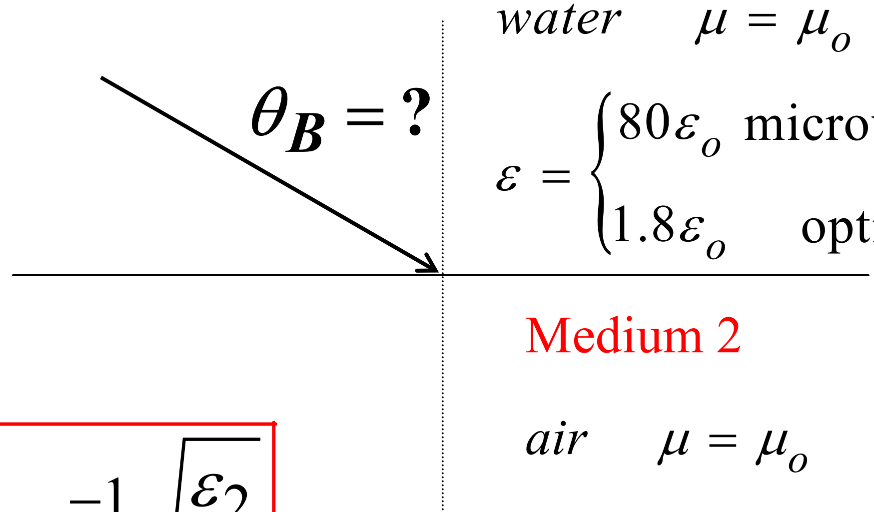
$$\sqrt{\varepsilon_1} \sin\theta_B = \sqrt{\varepsilon_2} \sin\theta_t$$

$$\theta_t = \sin^{-1}\left(\frac{\sin\theta_B}{\sqrt{\varepsilon_2}}\right) \cong \begin{cases} 6.38^\circ & \text{microwaves} \\ 36.7^\circ & \text{optical} \end{cases}$$

Medium 1

water $\mu = \mu_o$

$$\varepsilon = \begin{cases} 80\varepsilon_o & \text{microwaves} \\ 1.8\varepsilon_o & \text{optical} \end{cases}$$



Medium 2

air $\mu = \mu_o$

$\varepsilon = \varepsilon_o$

$$\theta_B = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\theta_B = \tan^{-1}\sqrt{\frac{1}{80}} \cong 6.38° \quad \text{microwaves}$$

$$\theta_B = \tan^{-1}\sqrt{\frac{1}{1.8}} \cong 36.7° \quad \text{optical}$$

At the Brewster angle

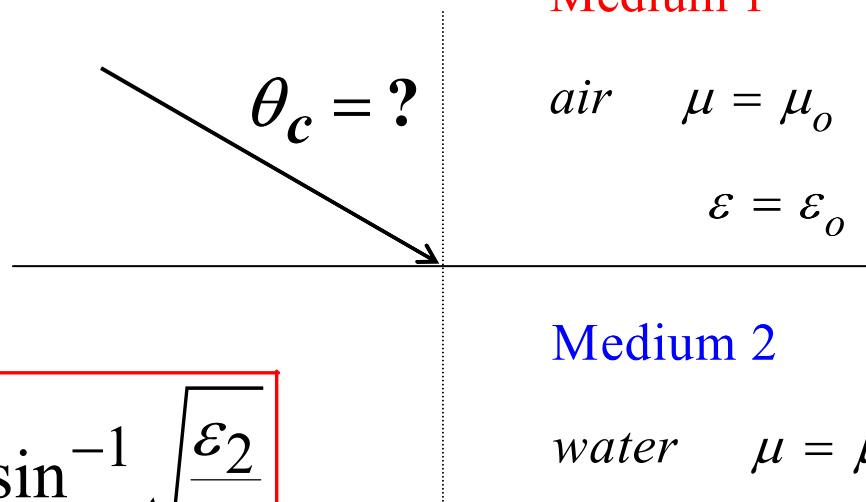
$$\theta_i + \theta_t = \theta_B + \theta_t = 90^\circ$$

$$\Rightarrow \theta_t \cong \begin{cases} 83.6^\circ & \text{microwaves} \\ 53.3^\circ & \text{optical} \end{cases}$$

Verification with Snell's law

$$\sqrt{\varepsilon_1}\sin\theta_B = \sqrt{\varepsilon_2}\sin\theta_t$$

$$\theta_t = \sin^{-1}\left(\frac{\sin\theta_B\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_o}}\right) \cong \begin{cases} 83.6^\circ & \text{microwaves} \\ 53.3^\circ & \text{optical} \end{cases}$$



Medium 1

air $\mu = \mu_o$

$\varepsilon = \varepsilon_o$

Medium 2

water $\mu = \mu_o$

$$\varepsilon = \begin{cases} 80\varepsilon_o & \text{microwaves} \\ 1.8\varepsilon_o & \text{optical} \end{cases}$$

$$\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

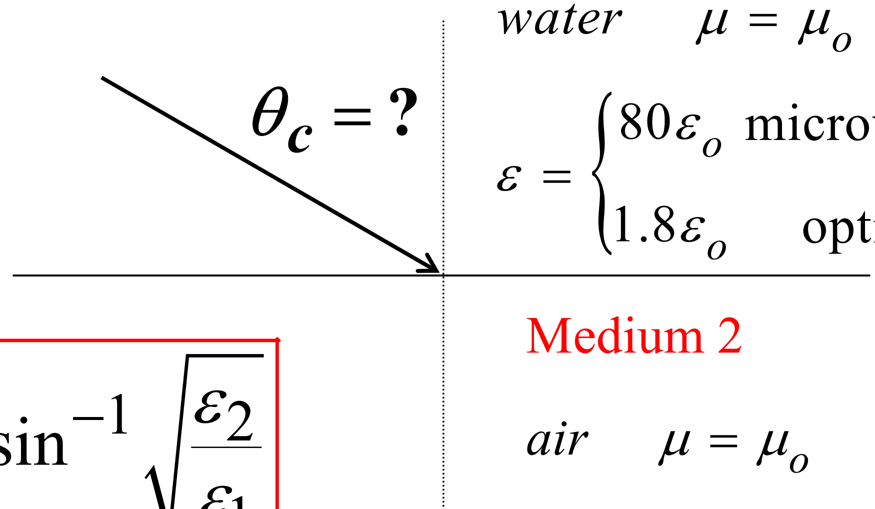
The total reflection angle does not exist since

$$\varepsilon_2 > \varepsilon_1$$

Medium 1

water $\mu = \mu_o$

$$\varepsilon = \begin{cases} 80\varepsilon_o & \text{microwaves} \\ 1.8\varepsilon_o & \text{optical} \end{cases}$$



$\theta_c = ?$

Medium 2

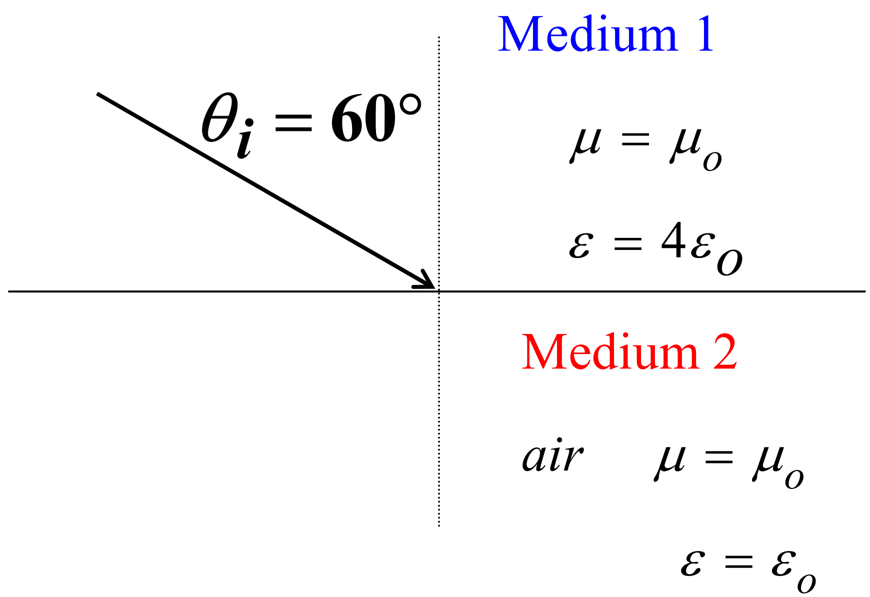
air $\mu = \mu_o$

$\varepsilon = \varepsilon_o$

$$\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\theta_c = \sin^{-1}\sqrt{\frac{1}{80}} \cong 6.4193° \quad \text{microwaves}$$

$$\theta_c = \sin^{-1}\sqrt{\frac{1}{1.8}} \cong 48.19° \quad \text{optical}$$



Consider a perpendicularly polarized wave.

- **Find the Brewster angle and the critical angle:**

$$\theta_B = \tan^{-1}\sqrt{\frac{1}{4}} = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.565°$$

$$\theta_c = \sin^{-1}\sqrt{\frac{1}{4}} = \sin^{-1}\left(\frac{1}{2}\right) = 30°$$

- **Find the components of the incident propagation vector and of the x-component of the transmitted propagation vector in terms of**

$$\beta_o = \omega\sqrt{\mu_o \varepsilon_o}$$

$$\beta_{ix} = \beta_1 \cos\theta_i = \omega\sqrt{\mu_o 4\varepsilon_o}\cos 60° = \frac{2\beta_o}{2} = \beta_o$$

$$\beta_{iz} = \beta_1 \sin\theta_i = \omega\sqrt{\mu_o 4\varepsilon_o}\sin 60° = 2\beta_o \frac{\sqrt{3}}{2} = \sqrt{3}\beta_o$$

$$\beta_{tx} = \sqrt{\beta_t^2 - \beta_{tz}^2} \qquad \beta_t = \beta_o \qquad \beta_{tz} = \beta_{iz} = \sqrt{3}\beta_o$$

$$\beta_{tx} = \underbrace{\pm}_{\substack{\text{choose} \\ \text{"}-\text{"}}} \sqrt{\beta_o^2 - 3\beta_o^2} = -j\sqrt{2}\beta_o = -j\alpha_t$$

- **In the second medium, find the distance at which the field strength is 1/e of that at the interface**

$$d = \frac{1}{\alpha_t} = \frac{1}{\sqrt{2}\beta_o}$$

- **What is the value of the magnitude of the reflection coefficient at the interface?**

The reflection coefficient is a complex quantity when the incident angle exceeds the critical angle. Because of total reflection we know that it must be

$$\left|\Gamma_{\perp}(E)\right| = 1$$

since the time-average power reflection coefficient is

$$R = \left|\Gamma_{\perp}(E)\right|^2 = 1$$

