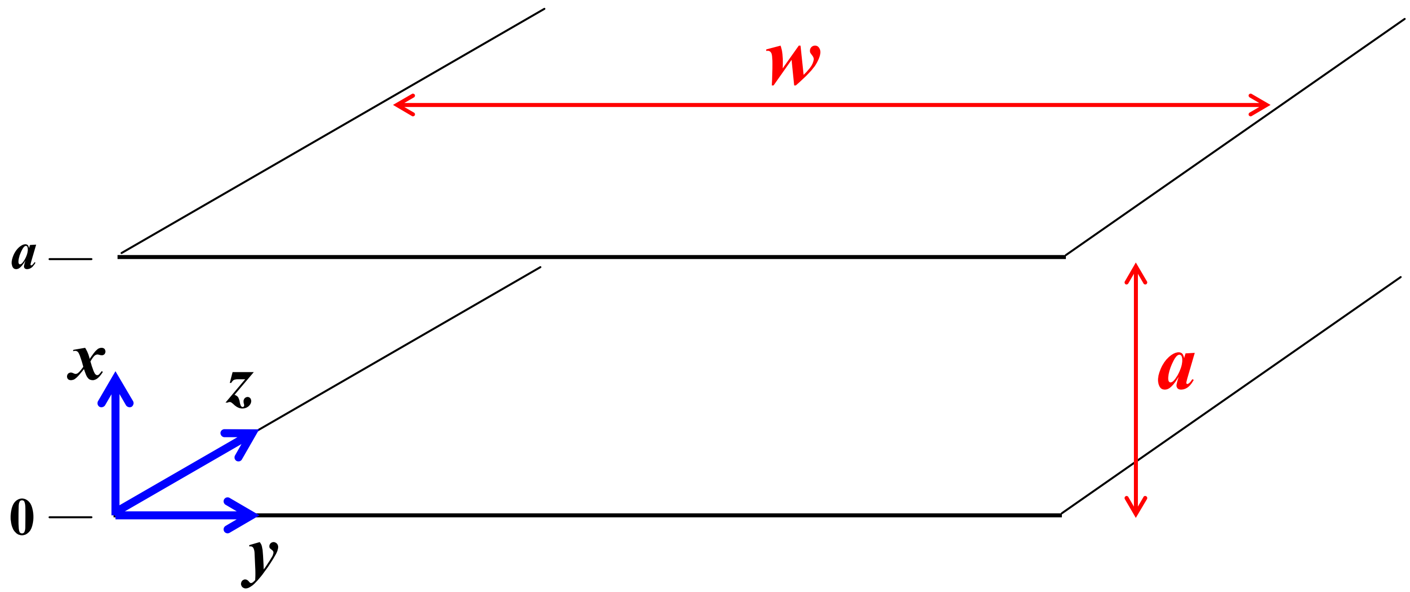
Parallel Plate Waveguide



Assume uniform waves along the *y*-direction $\Rightarrow \dfrac{\partial}{\partial y}(\ \ ) = 0$

Assume no fringing effects $\Rightarrow w >> a$

Propagation along the *z*-direction

Maxwell's equations

$$\nabla \times \vec{\mathbf{E}} = -j\omega\mu\,\vec{\mathbf{H}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{\boldsymbol{i}}_x & \hat{\boldsymbol{i}}_y & \hat{\boldsymbol{i}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{E}_x & \mathbf{E}_y & \mathbf{E}_z \end{bmatrix} \Rightarrow \begin{array}{ll} \cancel{\frac{\partial}{\partial y}\mathbf{E}_z} - \frac{\partial}{\partial z}\mathbf{E}_y = -j\omega\mu\mathbf{H}_x & \mathbf{(1)} \\ \frac{\partial}{\partial z}\mathbf{E}_x - \frac{\partial}{\partial x}\mathbf{E}_z = -j\omega\mu\mathbf{H}_y & \mathbf{(2)} \\ \frac{\partial}{\partial x}\mathbf{E}_y - \cancel{\frac{\partial}{\partial y}\mathbf{E}_x} = -j\omega\mu\mathbf{H}_z & \mathbf{(3)} \end{array}$$

$$\nabla \times \vec{\mathbf{H}} = j\omega\varepsilon\,\vec{\mathbf{E}}$$

$$\Downarrow$$

$$\det\begin{bmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathbf{H}_x & \mathbf{H}_y & \mathbf{H}_z \end{bmatrix} \Rightarrow \begin{aligned} &\cancel{\frac{\partial}{\partial y}}\mathbf{H}_z - \frac{\partial}{\partial z}\mathbf{H}_y = j\omega\varepsilon\mathbf{E}_x \qquad (4) \\ &\frac{\partial}{\partial z}\mathbf{H}_x - \frac{\partial}{\partial x}\mathbf{H}_z = j\omega\varepsilon\mathbf{E}_y \qquad (5) \\ &\frac{\partial}{\partial x}\mathbf{H}_y - \cancel{\frac{\partial}{\partial y}}\mathbf{H}_x = j\omega\varepsilon\mathbf{E}_z \qquad (6) \end{aligned}$$

From (1) & (2) & (5)

$$\frac{\partial}{\partial z}(1) \quad \Rightarrow \quad \frac{\partial^2}{\partial z^2}\mathbf{E}_y = j\omega\mu\frac{\partial}{\partial z}\mathbf{H}_x$$

$$\frac{\partial}{\partial x}(3) \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2}\mathbf{E}_y = -j\omega\mu\frac{\partial}{\partial x}\mathbf{H}_z$$

$$\frac{\partial^2}{\partial z^2}\mathbf{E}_y + \frac{\partial^2}{\partial x^2}\mathbf{E}_y = j\omega\mu\left(\frac{\partial}{\partial z}\mathbf{H}_x - \frac{\partial}{\partial x}\mathbf{H}_z\right) = -\omega^2\mu\varepsilon\mathbf{E}_y$$

$\Downarrow$

From (5) $\quad j\omega\varepsilon\mathbf{E}_y$

Wave equation for Transverse Electric (TE) modes

From (4) & (6) & (2)

$$\frac{\partial}{\partial z}(4) \quad \Rightarrow \quad \frac{\partial^2}{\partial z^2}\mathbf{H}_y = -j\omega\varepsilon \frac{\partial}{\partial z}\mathbf{E}_x$$

$$\frac{\partial}{\partial x}(6) \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2}\mathbf{H}_y = j\omega\varepsilon \frac{\partial}{\partial x}\mathbf{E}_z$$

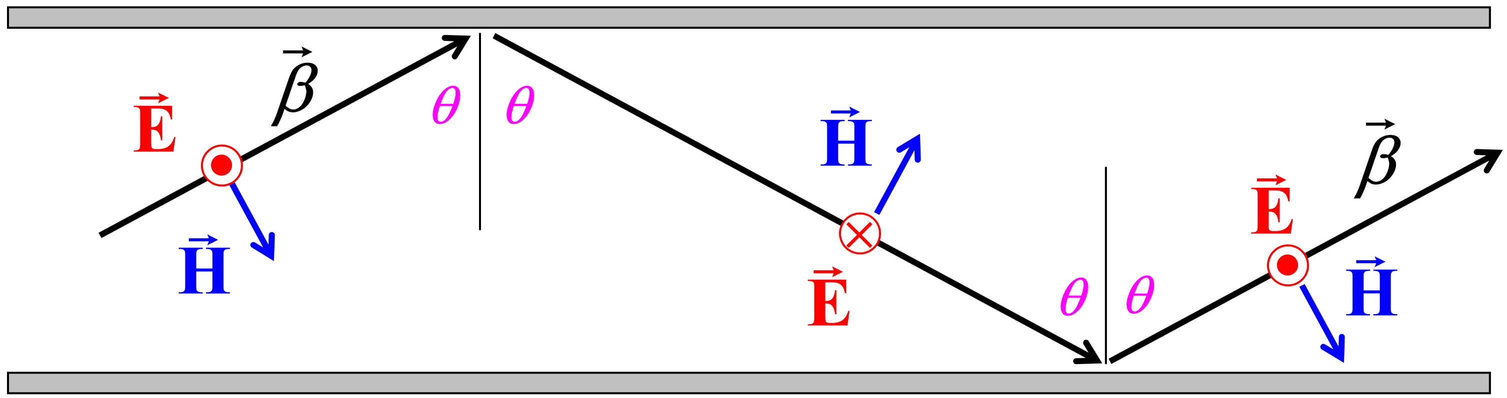
$$\frac{\partial^2}{\partial z^2}\mathbf{H}_y + \frac{\partial^2}{\partial x^2}\mathbf{H}_y = -j\omega\varepsilon\left(\frac{\partial}{\partial z}\mathbf{E}_x - \frac{\partial}{\partial x}\mathbf{E}_z\right) = -\omega^2\mu\varepsilon\mathbf{H}_y$$

$$\Downarrow$$

From (2) $-j\omega\mu\mathbf{H}_y$

Wave equation for Transverse Magnetic (TM) modes

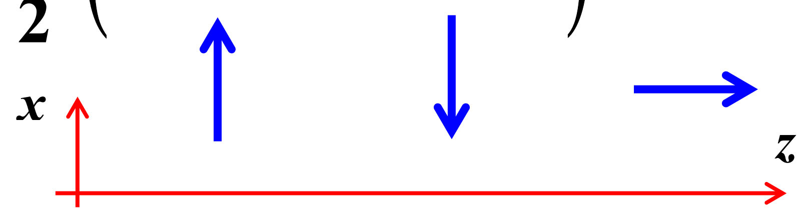
Transverse Electric (TE) modes



Boundary Conditions $\quad \mathbf{E}_y = 0 \quad \begin{cases} x = 0 \\ x = a \end{cases}$

This solution satisfies the boundary conditions:

$$\mathbf{E}_y = E_o \sin(\beta_x x) e^{-j\beta_z z} = j\frac{E_o}{2}\left(e^{-j\beta_x x} - e^{j\beta_x x}\right) e^{-j\beta_z z}$$



We have

$$\beta^2 = \frac{4\pi^2}{\lambda^2} = \beta_x^2 + \beta_z^2 = \omega^2 \mu\,\varepsilon$$

and from boundary conditions at the conductor plates

$$x = 0) \quad E_y = 0$$

$$x = a) \quad \sin(\beta_x a) = 0 \;\Rightarrow\; \beta_x a = m\pi$$

$$m = 1,\ 2,\ 3 \ldots$$

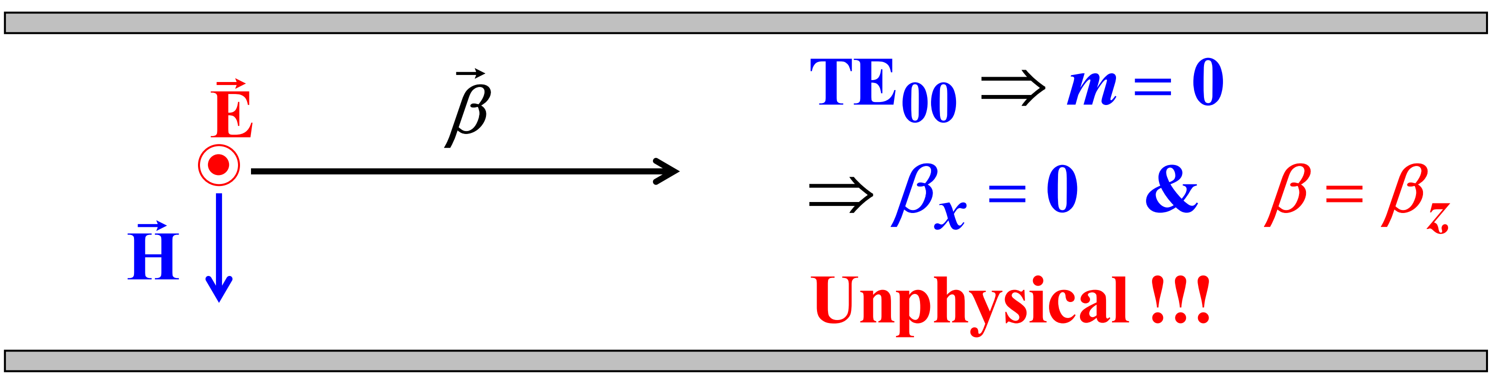
$$\beta_x = \beta \cos\theta = \frac{m\pi}{a}$$

$$\beta_z = \beta \sin\theta = \sqrt{\omega^2 \mu\,\varepsilon - \left(\frac{m\pi}{a}\right)^2} = \omega\sqrt{\mu\,\varepsilon}\left(1 - \left(\frac{m\lambda}{2a}\right)^2\right)^{1/2}$$

For each possible index m we have a mode of propagation. Modes are labeled TE_{10} , TE_{20} , TE_{30} , ….

The first index gives the periodicity (number of half sinusoidal oscillations) between the plates, along the x-direction. The second index is zero to indicate uniform solution along the y-direction.

Note that the solution $m = 0$ (or mode TE_{00}) is not acceptable, because it would require a field configuration with uniform electric field tangent to the metal plates. This is an unphysical boundary condition, which is possible only for the case of trivial solution of zero field everywhere.



A mode can propagate only if the frequency is sufficiently high, so that $\beta_z > 0$.

We have the cut-off condition when

$$\beta = \beta_x = \frac{m\pi}{a} = \frac{2\pi}{\lambda_c} \Rightarrow \lambda_c = \frac{2a}{m}$$

$$\Rightarrow \beta_z = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2} = \omega\sqrt{\mu\varepsilon}\left(1 - \left(\frac{m\lambda}{2a}\right)^2\right)^{1/2} = 0$$

$$f_c = \frac{v_p}{\lambda_c} = \frac{m}{2a\sqrt{\mu\varepsilon}}$$ Cut - off frequency for mode m

Exactly at cut-off the wave would bounce between the plates, without propagation along the wave guide axis.

When the frequency is below the cut-off value

$$f < f_c \Rightarrow \lambda > \lambda_c = \frac{2a}{m}$$

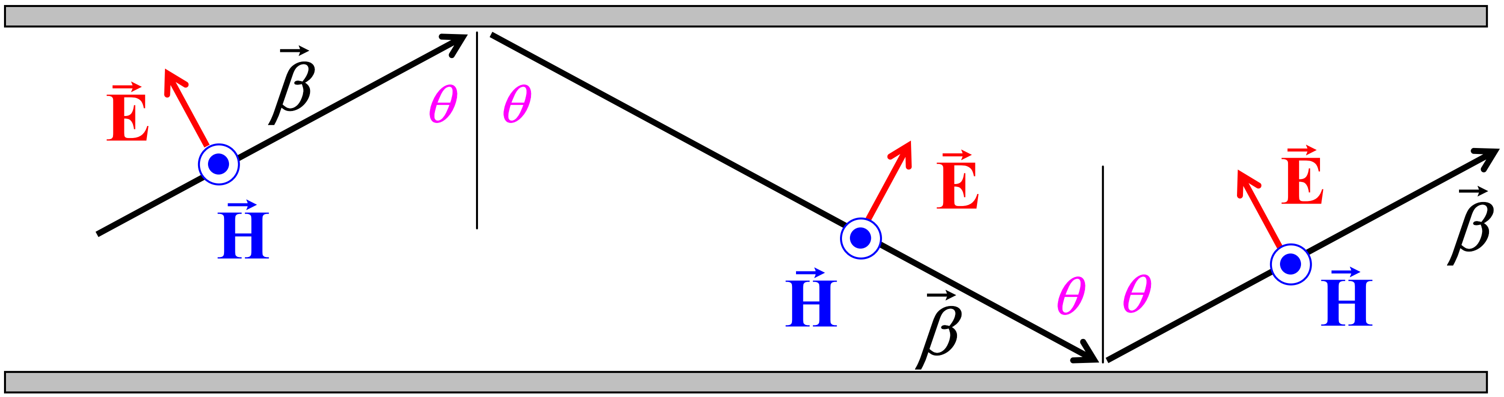
$$\beta_z = \pm\sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2} = \pm\omega\sqrt{\mu\varepsilon}\left(1 - \underbrace{\left(\frac{m\lambda}{2a}\right)^2}_{>1}\right)^{1/2}$$

$$= \quad \pm j\,\omega\sqrt{\mu\varepsilon}\left(\left(\frac{m\lambda}{2a}\right)^2 - 1\right)^{1/2}$$

$$\Rightarrow \quad \beta_z = -j\alpha \quad \Rightarrow e^{-j(-j\alpha)z} = e^{-\alpha z}$$

The mode attenuates entering the guide as an evanescent wave.

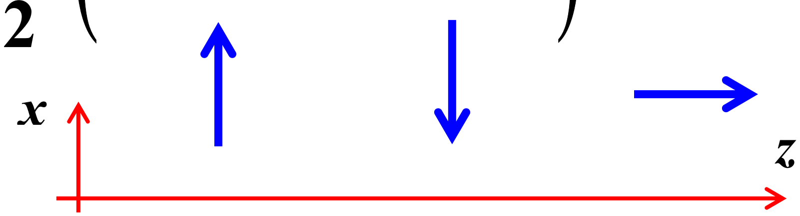
Transverse Magnetic (TM) modes



The magnetic field can be tangent to the conductor plates. In fact, it is maximum at the plates, since the reflection coefficient is $\Gamma_H = 1$.

The solution is of the form:

$$\mathbf{H}_y = H_o \cos(\beta_x x)\, e^{-j\beta_z z} = \frac{H_o}{2}\left(e^{-j\beta_x x} + e^{j\beta_x x}\right) e^{-j\beta_z z}$$



At the metal plates

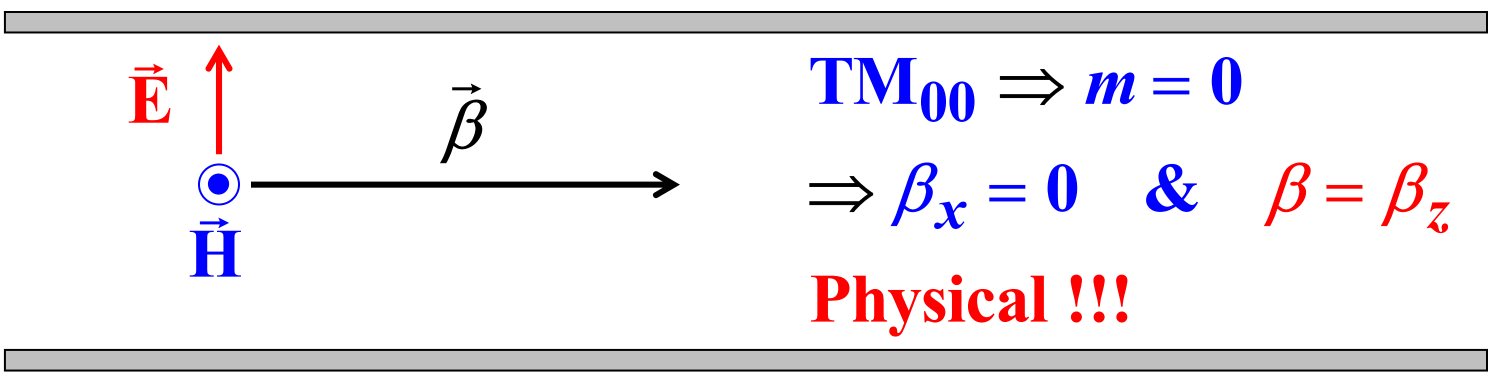
$$x = 0) \quad H_y = H_o$$

$$x = a) \quad \cos(\beta_x a) = 1 \;\Rightarrow\; \beta_x a = m\pi$$

$$m = 0,\, 1,\, 2,\, 3 \ldots$$

Modes are labeled TM_{00} , TM_{10} , TM_{20} , TM_{30} , …

Note that the solution $m = 0$ (or mode TM_{00}) is acceptable, because the magnetic field can be uniform and tangent to the metal plates.



The TM_{00} mode is like a portion of a uniform plane wave sliding between the plates of the waveguide.

Both the electric and the magnetic field are transverse (normal to the guide axis) therefore this mode is usually known as Transverse Electro Magnetic mode (TEM). For this mode we have

$$\beta = \beta_z = \frac{2\pi}{\lambda} \quad \Rightarrow \beta_x = \frac{2\pi}{\lambda_c} = 0 \Rightarrow \lambda_c \rightarrow \infty$$

$$f_c = \frac{v_p}{\lambda_c} = 0 \quad \text{Cut - off frequency for TEM mode}$$

The TEM mode is the fundamental mode. It can propagate at any frequency.

All other TM modes have the same cut-off frequency condition as the TE modes with identical indices.

The apparent wavelength along the guide axis is also called the guide wavelength

$$\lambda_g = \lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{\beta \sin\theta}$$

$$\text{Since}: \beta_x = \beta \cos\theta = \frac{m\pi}{a} = \frac{2\pi}{\lambda_c} = \frac{2\pi}{\lambda}\frac{\lambda}{\lambda_c}$$

$$\cos\theta = \frac{\lambda}{\lambda_c} = \frac{f_c}{f} \quad \Rightarrow \quad \sin\theta = \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (\lambda / \lambda_c)^2}} = \frac{\lambda}{\sqrt{1 - (f_c / f)^2}}$$

There is a corresponding apparent velocity along the guide axis, or guide phase velocity

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{\omega}{\beta \sin\theta}$$

$$v_{pz} = \frac{v_p}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{v_p}{\sqrt{1-(f_c/f)^2}}$$

The expressions for guide wavelength and guide velocity are also identical for TE and TM modes.

Consider a TE wave with electric field amplitude E_o. The total amplitude of the magnetic field is

$$H_o = \frac{E_o}{\eta}$$

The magnetic field has two components with amplitude

$$|\mathbf{H}_x| = |H_o| \sin\theta = \frac{|E_o|}{\eta} \sin\theta = \frac{|E_o|}{\eta} \frac{\lambda}{\lambda_g}$$

$$\text{since}: \quad \lambda_g = \frac{2\pi}{\beta \sin\theta} = \frac{\lambda}{\sin\theta}$$

$$|\mathbf{H}_z| = |H_o| \cos\theta = \frac{|E_o|}{\eta} \cos\theta = \frac{|E_o|}{\eta} \frac{\lambda}{\lambda_c}$$

$$\text{since}: \quad \lambda_c = \frac{2a}{m} = \frac{\lambda}{\cos\theta}$$

Consider a TM wave with magnetic field amplitude H_o. The total amplitude of the electric field is

$$E_o = \eta H_o$$

The electric field has two components with amplitude

$$|\mathbf{E}_x| = |E_o| \sin\theta = \eta |H_0| \sin\theta = \eta |H_o| \frac{\lambda}{\lambda_g}$$

$$\text{since :} \quad \lambda_g = \frac{2\pi}{\beta \sin\theta} = \frac{\lambda}{\sin\theta}$$

$$|\mathbf{E}_z| = |E_o| \cos\theta = \eta |H_o| \cos\theta = \eta |H_o| \frac{\lambda}{\lambda_c}$$

$$\text{since :} \quad \lambda_c = \frac{2a}{m} = \frac{\lambda}{\cos\theta}$$

The x–component of the magnetic field for the TE wave is associated with the wave moving along the z–direction (axis of the waveguide). The guide impedance for the TE modes is defined as

$$\eta_{g_{TE}} = \eta \frac{\lambda_g}{\lambda} = \eta \frac{1}{\sqrt{1-(\lambda/\lambda_c)^2}} = \eta \frac{1}{\sqrt{1-(f_c/f)^2}}$$

The x–component of the electric field for the TM wave is associated with the wave moving along the z–direction (axis of the waveguide). The guide impedance for the TM modes is defined as

$$\eta_{g_{TM}} = \eta \frac{\lambda}{\lambda_g} = \eta\sqrt{1-(\lambda/\lambda_c)^2} = \eta\sqrt{1-(f_c/f)^2}$$

If there is a discontinuity along the guide axis (e.g., a change in dielectric medium), one can use transmission line theory to analyze the mode behavior individually in terms of transmission and reflection. Sections of the guide can be replaced by a transmission line, with the guide impedance as the characteristic impedance.

Note that the guide impedance is a function of frequency for all modes, except for the fundamental TEM mode

$$f_c(TEM) = 0 \quad \Rightarrow \quad \eta_{g_{TEM}} = \eta\sqrt{1-(0/f)^2} = \eta$$

The reflection coefficient at a discontinuity is of the usual form

$$\Gamma = \frac{\eta_{g2} - \eta_{g1}}{\eta_{g2} + \eta_{g1}}$$

The power reflection coefficient is again $|\Gamma|^2$ and the power transmission coefficient is $1-|\Gamma|^2$.

The phasor fields for TE modes are summarized as follows

Electric Field: a single transverse component

$$\vec{\mathbf{E}} = E_o \sin(\beta_x x)\, e^{-j\beta_z z}\, \hat{i}_y = E_o \sin\left(\frac{m\pi}{a} x\right) e^{-j\beta_z z}\, \hat{i}_y$$

Magnetic Field: two components, obtained from Faraday's law:

$$\nabla \times \vec{\mathbf{E}}_y = -\frac{\partial}{\partial z}\mathbf{E}_y\, \hat{i}_x + \frac{\partial}{\partial x}\mathbf{E}_y\, \hat{i}_z = -j\omega\mu\left(H_x\, \hat{i}_x + H_z\, \hat{i}_z\right)$$

$$\Rightarrow \vec{\mathbf{H}} = -E_o \frac{\beta_z}{\omega\mu} \sin\left(\frac{m\pi}{a} x\right) e^{-j\beta_z z}\, \hat{i}_x + jE_o \frac{\beta_x}{\omega\mu} \cos\left(\frac{m\pi}{a} x\right) e^{-j\beta_z z}\, \hat{i}_z$$

The following relationships are useful to introduce the medium impedance in the TE field expressions above

$$\frac{\beta_z}{\omega\mu} = \frac{\omega\sqrt{\mu\varepsilon}\,\sin\theta}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{\lambda}{\lambda_g} = \frac{\lambda}{\eta\,\lambda_g} = \frac{1}{\eta_{g_{TE}}}$$

$$\frac{\beta_x}{\omega\mu} = \frac{\omega\sqrt{\mu\varepsilon}\,\cos\theta}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{\lambda}{\lambda_c} = \frac{\lambda}{\eta\,\lambda_c}$$

Note once again that there is no allowed solution for *m* = 0 in the case of TE modes. The first allowed TE mode is the TE_{10}.

The phasor fields for TM modes are summarized as follows

Magnetic Field: a single transverse component

$$\vec{H}_y = H_o \cos(\beta_x x) e^{-j\beta_z z}\, \hat{i}_y = H_o \cos\left(\frac{m\pi}{a} x\right) e^{-j\beta_z z}\, \hat{i}_y$$

Electric Field: two components, obtained from Ampere's law:

$$\nabla \times \vec{H}_y = -\frac{\partial}{\partial z} H_y\, \hat{i}_x + \frac{\partial}{\partial x} H_y\, \hat{i}_z = j\omega\varepsilon (E_x\, \hat{i}_x + E_z\, \hat{i}_z)$$

$$\Rightarrow \vec{E} = H_o \frac{\beta_z}{\omega\varepsilon} \cos\left(\frac{m\pi}{a} x\right) e^{-j\beta_z z}\, \hat{i}_x$$

$$+\, jH_o \frac{\beta_x}{\omega\varepsilon} \sin\left(\frac{m\pi}{a} x\right) e^{-j\beta_z z}\, \hat{i}_z$$

The following relationships are useful to introduce the medium impedance in the TM field expressions above

$$\frac{\beta_z}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}\,\sin\theta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\lambda}{\lambda_g} = \eta\,\frac{\lambda}{\lambda_g} = \eta_{g_{TM}}$$

$$\frac{\beta_x}{\omega\varepsilon} = \frac{\omega\sqrt{\mu\varepsilon}\,\cos\theta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\lambda}{\lambda_c} = \eta\,\frac{\lambda}{\lambda_c}$$

The field expressions simplified for the TEM mode resemble a uniform plane wave propagating along the axis of the guide

$$\vec{\mathbf{H}}_y = H_o\, e^{-j\beta_z z}\, \hat{i}_y$$

$$\vec{\mathbf{E}}_x = \eta H_o\, e^{-j\beta_z z}\, \hat{i}_x = E_o\, e^{-j\beta_z z}\, \hat{i}_x$$

Remember, the TM_{00} or TEM mode is the fundamental mode.

Wave Dispersion

A plane wave by itself does not carry information. For transmission of information it is necessary to have a frequency spectrum of finite size, as obtained by modulation of a wave, for instance.

Information does not travel at the guide phase velocity, but it propagates according to the group velocity

guide phase velocity $$v_{pz} = \frac{\omega}{\beta_z}$$

group velocity $$v_g = \frac{d\omega}{d\beta_z}$$

To illustrate the nature of the group velocity, consider the simple case of an amplitude modulated signal (assume $\omega >> \Delta\omega$)

$$E_y(t) = E_o\left(1 + m\cos\left(\Delta\omega \cdot t\right)\right)\cos\left(\omega_o t\right)$$

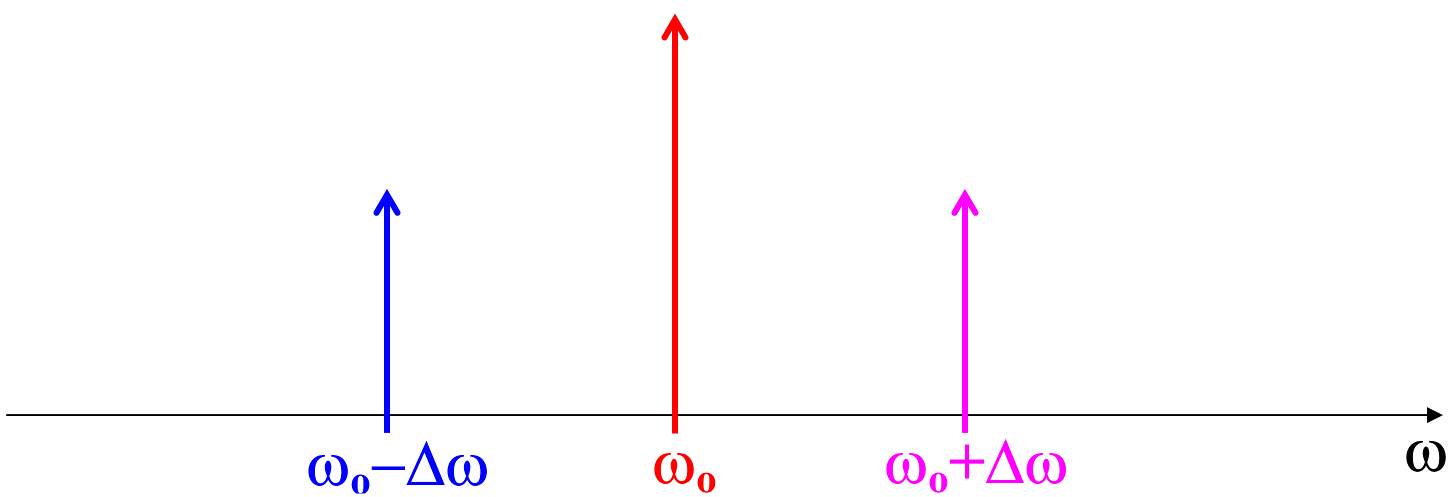
This signal has three components

$$E_y(t) = E_o \cos(\omega_o t) + m E_o \cos(\omega_o t) \cos(\Delta\omega \cdot t)$$

$$= E_o \cos(\omega_o t)$$

$$+ \frac{m}{2} E_o \cos(\omega_o t - \Delta\omega \cdot t)$$

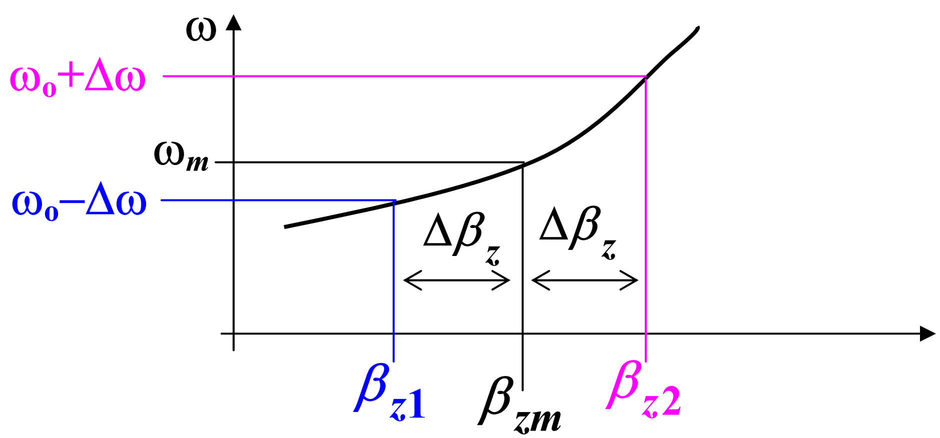
$$+ \frac{m}{2} E_o \cos(\omega_o t + \Delta\omega \cdot t)$$



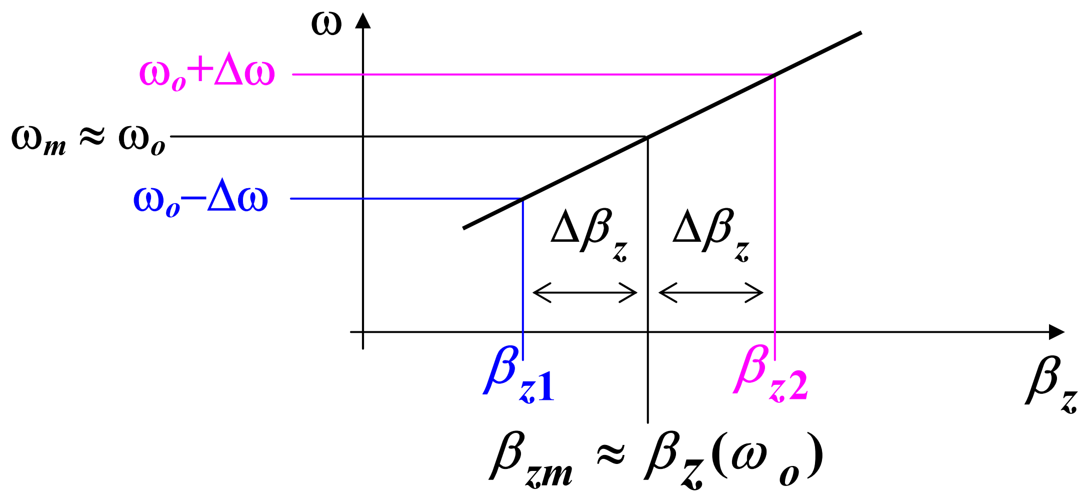
The line at angular frequency ω_o is the carrier. The modulation information is contained in the two side frequency lines at $\omega_o - \Delta\omega$ and $\omega_o + \Delta\omega$.

Now, consider an amplitude modulated wave propagating in a parallel plate wave guide. The z–components of the propagation factor depend on frequency and are different for the two side frequencies. In general, we have

$$\beta_{z1} = \beta_{zm} - \Delta\beta_z \qquad \beta_{z2} = \beta_{zm} + \Delta\beta_z$$



The dispersion relation $\beta(\omega)$ is approximately linear when $\Delta\omega \ll \omega_o$



Under this assumption, we can write

$$E_y(z,t) = E_o \cos(\omega_o t - \beta_z z)$$

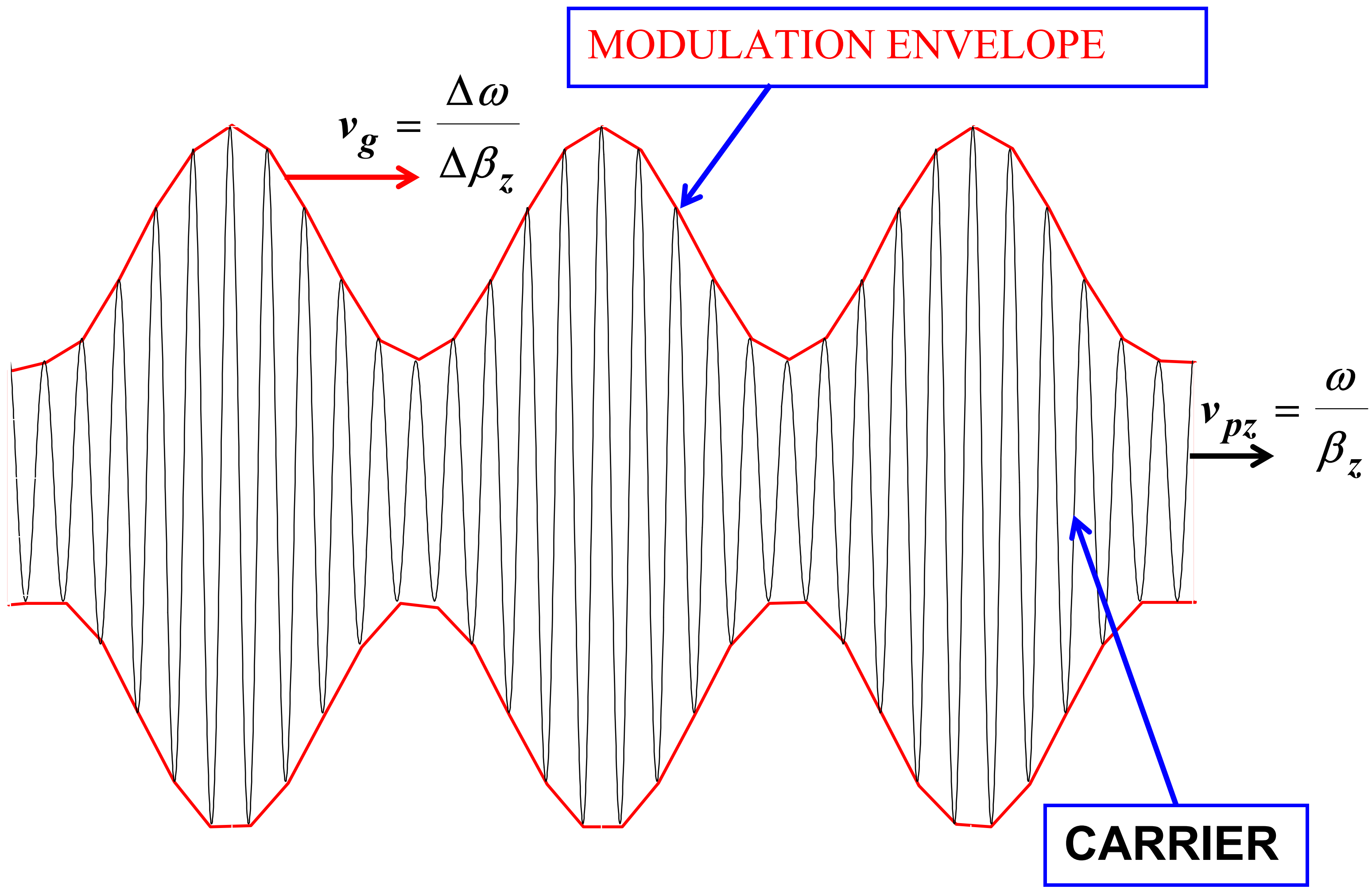
$$+ \frac{m}{2} E_o \cos\left[(\omega_o - \Delta\omega) t - (\beta_z - \Delta\beta_z) z\right]$$

$$+ \frac{m}{2} E_o \cos\left[(\omega_o + \Delta\omega) t - (\beta_z + \Delta\beta_z) z\right]$$

$$E_y(z,t) = E_o \cos(\omega_o t - \beta_z z)$$
$$+ \frac{m}{2} E_o \cos[(\omega_o t - \beta_z z) - (\Delta\omega \cdot t - \Delta\beta_z z)]$$
$$+ \frac{m}{2} E_o \cos[(\omega_o t - \beta_z z) + (\Delta\omega \cdot t - \Delta\beta_z z)]$$
$$= E_o \cos(\omega_o t - \beta_z z)$$
$$+ m E_o \cos(\omega_o t - \beta_z z) \cos(\Delta\omega \cdot t - \Delta\beta_z z)$$
$$= \underbrace{E_o [1 + m \cos(\Delta\omega \cdot t - \Delta\beta_z z)]}_{\text{modulated amplitude}} \cos(\omega_o t - \beta_z z)$$

The modulation envelope travels at the group velocity

$$v_g = \Delta\omega / \Delta\beta_z$$


MODULATION ENVELOPE
$v_g = \dfrac{\Delta\omega}{\Delta\beta_z}$
$v_{pz} = \dfrac{\omega}{\beta_z}$
CARRIER

For the parallel plate wave guide

$$\beta_z = \omega\sqrt{\mu\,\varepsilon}\left(1-\left(\frac{\lambda}{\lambda_c}\right)^2\right)^{1/2} = \omega\sqrt{\mu\,\varepsilon}\left(1-\left(\frac{f_c}{f}\right)^2\right)^{1/2}$$

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{v_p}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{v_p}{\sqrt{1-(f_c/f)^2}}$$

$$v_g = \frac{d\omega}{d\beta_z} = v_p\sqrt{1-(\lambda/\lambda_c)^2} = v_p\sqrt{1-(f_c/f)^2}$$

$$\Rightarrow \quad v_{pz}\cdot v_g = v_p^2$$

Since $v_{pz} \geq v_p \quad \Rightarrow \quad v_g \leq v_p$

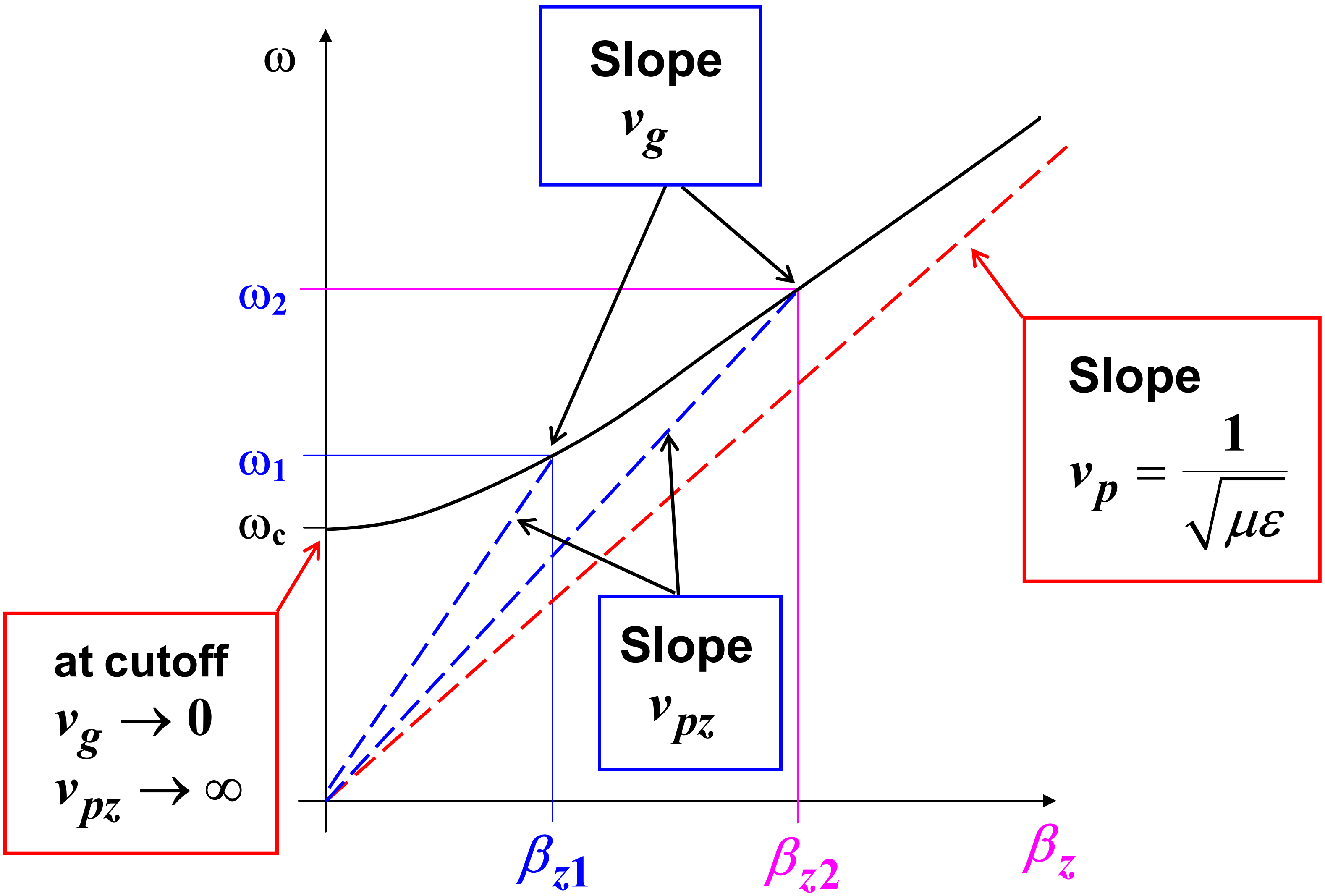
Information travels at the group velocity, which is always less than the corresponding phase velocity in the given medium.

The group and phase velocities for each mode propagating in the wave guide are frequency–dependent. This means that frequency components of a broadband signal travel at different speed and change their phase relationship as they propagate along the wave guide. The group and phase velocities of the modes are also mode–dependent. This means that if a signal is distributed over a number of different modes, the components spread out over time during propagation.

This phenomenon is called dispersion. Wave guides are in general dispersive media.

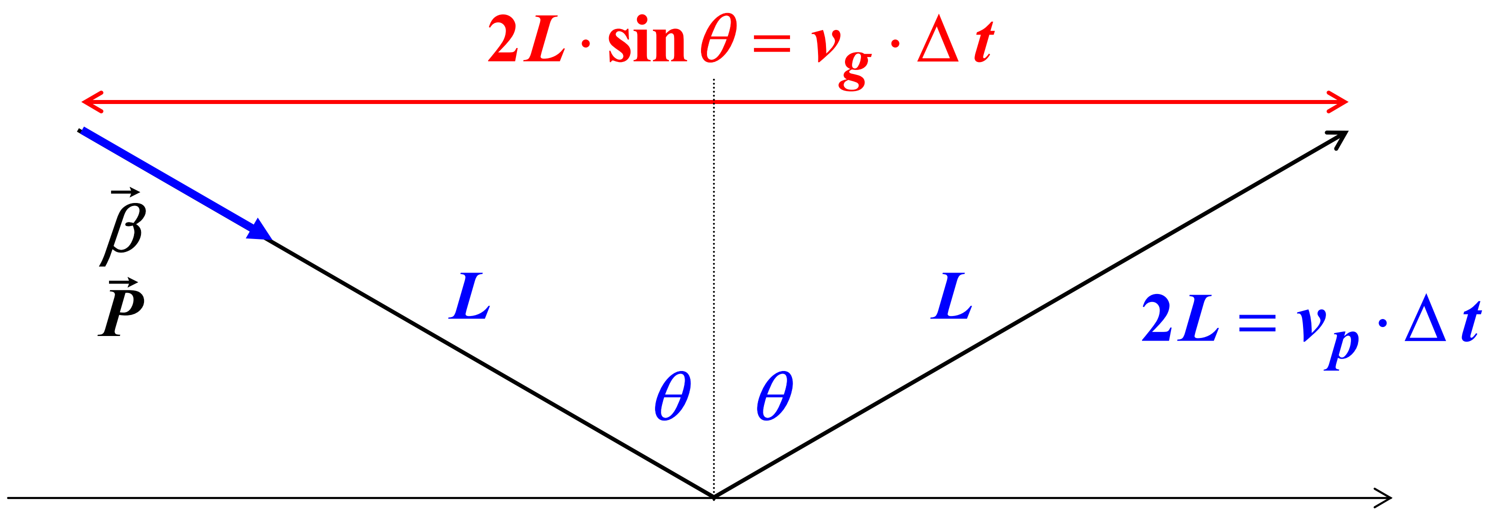
Note: For the fundamental TEM mode in parallel plate wave guide

$$f_c = 0 \quad \Rightarrow \quad v_{pz} = v_p = v_g \quad \Rightarrow \quad \text{no dispersion}$$



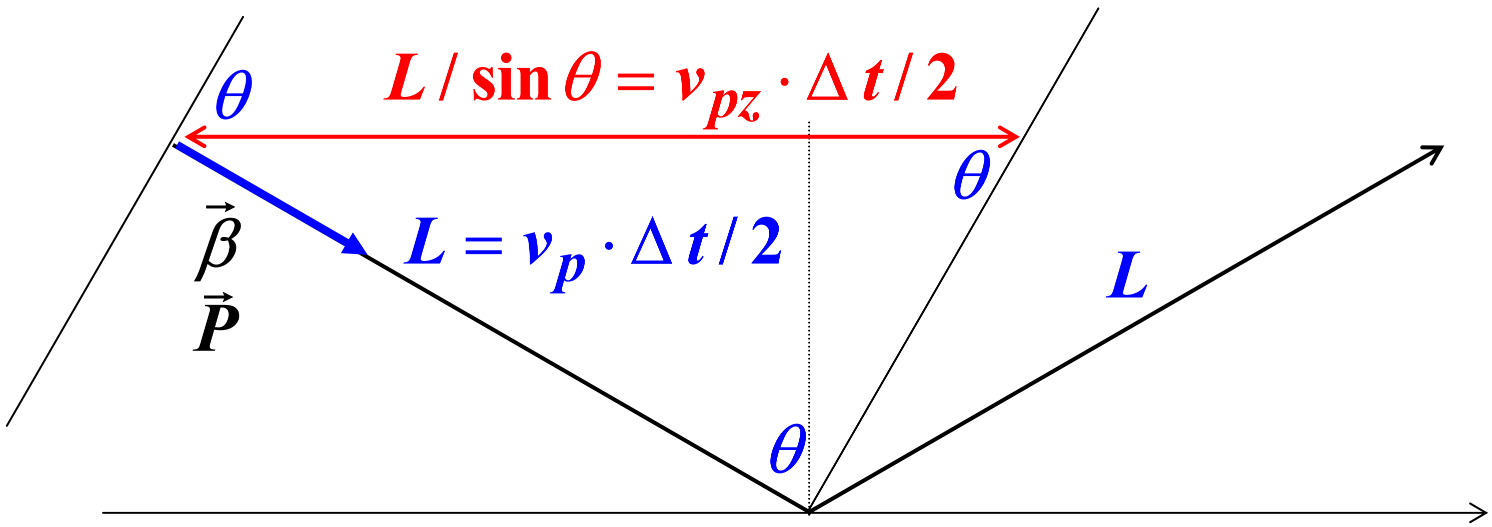
Dispersion diagram

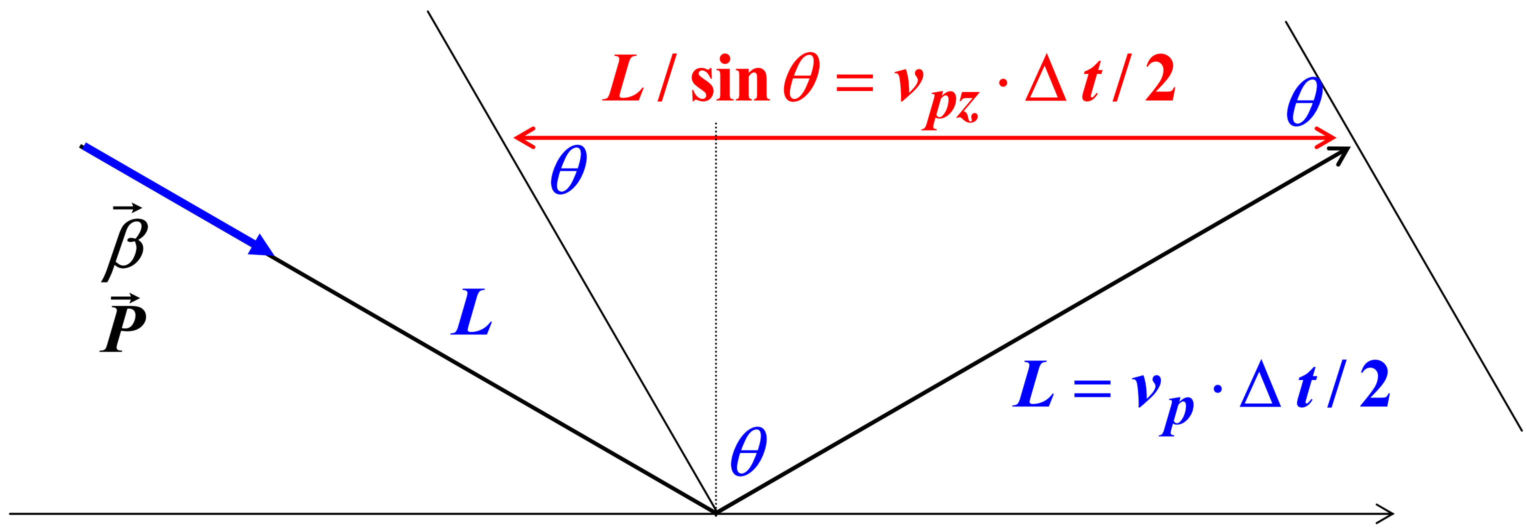
The power flow follows the Poynting vector, with the same direction as the propagation vector. The group velocity accounts for the effective motion of the power flow in the direction parallel to the axis of the wave guide.



$$\Delta t = \frac{2L \cdot \sin\theta}{v_g} = \frac{2L}{v_p} \quad \Rightarrow \quad v_g = v_p \sin\theta$$

The guide phase velocity corresponds to the apparent motion illustrated by the following diagrams





Therefore, we obtain for the guide phase velocity

$$\Delta t = \frac{2L}{v_{pz} \cdot \sin\theta} = \frac{2L}{v_p} \quad \Rightarrow \quad v_{pz} = \frac{v_p}{\sin\theta}$$

From the results above, we have again

$$v_{pz} \geq v_p$$

$$v_g \leq v_p$$

$$v_{pz} \cdot v_g = \frac{v_p}{\sin\theta} v_p \sin\theta = v_p^2$$