Plane Waves in Arbitrary Directions

For a uniform plane wave with general orientation, the direction of propagation is identified by the propagation vector, normal to the phase planes

$$\vec{\beta} = \beta_x \hat{i}_x + \beta_y \hat{i}_y + \beta_z \hat{i}_z$$

Considering the position vector

$$\vec{\mathbf{r}} = x\,\hat{i}_x + y\,\hat{i}_y + z\,\hat{i}_z$$

we have the scalar dot product

$$\vec{\beta} \cdot \vec{\mathbf{r}} = \left(\beta_x \hat{i}_x + \beta_y \hat{i}_y + \beta_z \hat{i}_z\right) \cdot \left(x\,\hat{i}_x + y\,\hat{i}_y + z\,\hat{i}_z\right) =$$

$$= \beta_x\, x + \beta_y\, y + \beta_z\, z$$

The electromagnetic fields of a plane wave propagating along a general direction are on the phase plane perpendicular to the propagation vector and can be written as

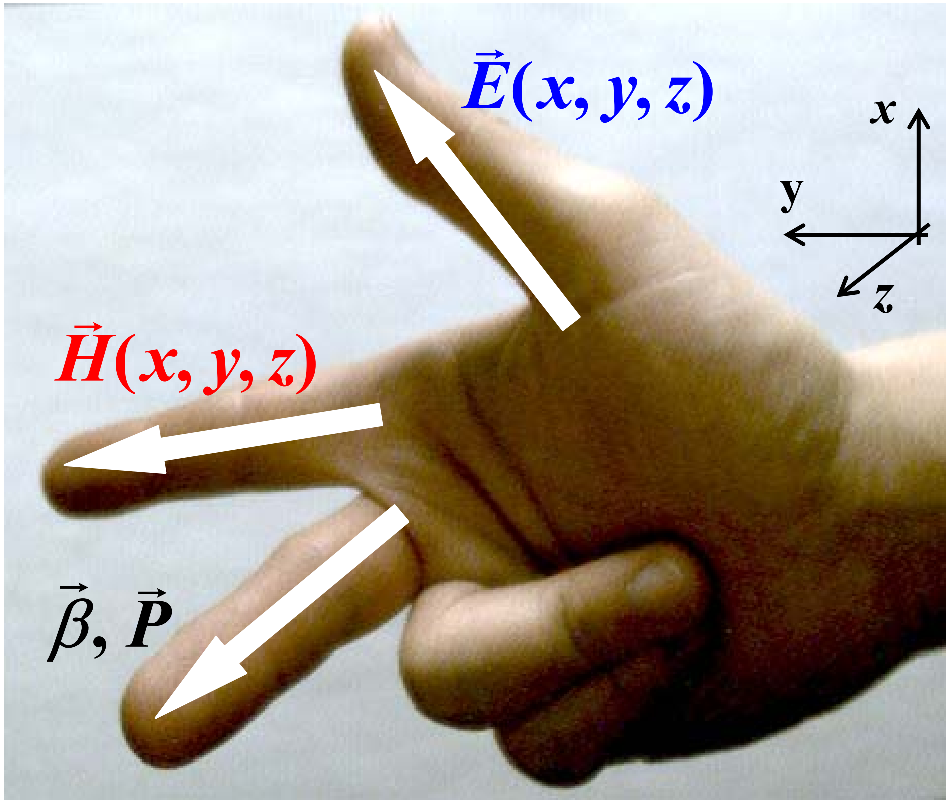
$$\vec{E}(\vec{\mathbf{r}},t) = \vec{E}_o \cos\left(\omega t - \vec{\beta}\cdot\vec{\mathbf{r}} + \varphi_o\right)$$
$$= \vec{E}_o \cos\left(\omega t - \beta_x x - \beta_y y - \beta_z z + \varphi_o\right)$$

$$\vec{H}(\vec{\mathbf{r}},t) = \vec{H}_o \cos\left(\omega t - \vec{\beta}\cdot\vec{\mathbf{r}} + \varphi_o\right)$$
$$= \vec{H}_o \cos\left(\omega t - \beta_x x - \beta_y y - \beta_z z + \varphi_o\right)$$

We have assumed propagation in a dielectric by giving the same phase to the fields. In addition

$$\left|\vec{H}_o\right| = \frac{\left|\vec{E}_o\right|}{\eta} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} \quad \textbf{(dielectric)}$$

The fields are perpendicular to each other and to the propagation vector according to the right hand rule



The propagation vector is also parallel to the Poynting vector.

$$\vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t) = \vec{P}(t) \propto \vec{\beta}$$

The orthogonality of the vectors can be expressed mathematically by the following dot products

$$\vec{E}_o \cdot \vec{H}_o = \vec{E}_o \cdot \vec{\beta} = \vec{H}_o \cdot \vec{\beta} = 0$$

and cross products

$$\vec{H}_o = \vec{i}_\beta \times \frac{\vec{E}_o}{\eta} \qquad \vec{E}_o = -\vec{i}_\beta \times \eta \vec{H}_o$$

We have

$$\vec{H}_o = \vec{i}_\beta \times \frac{\vec{E}_o}{\eta} = \frac{\vec{\beta}}{|\beta|} \times \frac{\vec{E}_o}{\sqrt{\mu / \varepsilon}} = \frac{\vec{\beta}}{\omega\sqrt{\mu\varepsilon}} \times \frac{\vec{E}_o}{\sqrt{\mu / \varepsilon}}$$

$$\vec{H}_o = \frac{\vec{\beta} \times \vec{E}_o}{\omega\mu}$$

$$\vec{E}_o = -\vec{i}_\beta \times \eta \vec{H}_o = -\frac{\vec{\beta}}{|\vec{\beta}|} \times \sqrt{\frac{\mu}{\varepsilon}}\, \vec{H}_o = -\frac{\vec{\beta}}{\omega\sqrt{\mu\varepsilon}} \times \sqrt{\frac{\mu}{\varepsilon}}\, \vec{H}_o$$

$$\vec{E}_o = -\frac{\vec{\beta} \times \vec{H}_o}{\omega\, \varepsilon}$$

Since the propagation vector is related to the wavelength and the phase velocity as

$$|\vec{\beta}| = \frac{2\pi}{\lambda} = \frac{\omega}{|\vec{v}_p|}$$

for each direction corresponding to components of the propagation vector we can define apparent wavelengths and apparent velocities

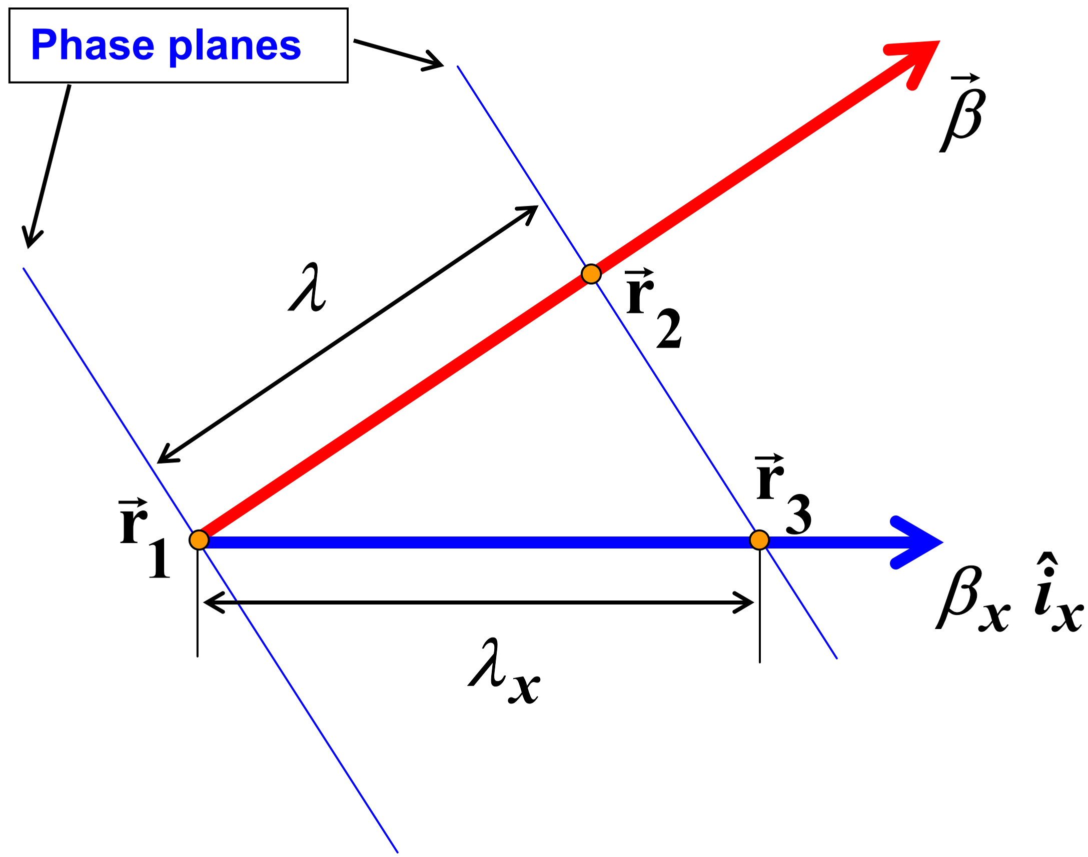
$$\lambda_x = \frac{2\pi}{\beta_x} \qquad \lambda_y = \frac{2\pi}{\beta_y} \qquad \lambda_z = \frac{2\pi}{\beta_z}$$

$$v_{px} = \frac{\omega}{\beta_x} \qquad v_{py} = \frac{\omega}{\beta_y} \qquad v_{pz} = \frac{\omega}{\beta_z}$$

The apparent quantities are related to the actual ones as

$$\frac{1}{\lambda^2} = \frac{\beta^2}{4\pi^2} = \frac{\beta_x^2}{4\pi^2} + \frac{\beta_y^2}{4\pi^2} + \frac{\beta_z^2}{4\pi^2} = \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} + \frac{1}{\lambda_z^2}$$

$$\frac{1}{v_p^2} = \frac{\beta^2}{\omega^2} = \frac{\beta_x^2}{\omega^2} + \frac{\beta_y^2}{\omega^2} + \frac{\beta_z^2}{\omega^2} = \frac{1}{v_{px}^2} + \frac{1}{v_{py}^2} + \frac{1}{v_{pz}^2}$$



An apparent wavelength is greater than the actual one, since it is measured along a direction which is not normal to the parallel phase planes.

An apparent velocity is greater than the actual velocity since it seemingly connects longer distances during the same time. With reference to the previous figure

$$\left(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1\right) = \vec{v}_p \, t_1$$

$$\left(\vec{\mathbf{r}}_3 - \vec{\mathbf{r}}_1\right) = v_{px} \, \vec{i}_x \, t_1$$

Since

$$\left|\left(\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1\right)\right| < \left|\left(\vec{\mathbf{r}}_3 - \vec{\mathbf{r}}_1\right)\right| \quad \Rightarrow \quad \left|\vec{v}_p\right| < v_{px}$$

The apparent velocity always exceeds the phase velocity. However, we will see later that relativity laws are not violated.

If one considers a direction parallel to the phase planes, the apparent velocity is even infinite!

Phasor notation

$$\vec{E}(\vec{r},t) = \vec{E}_o \cos\left(\omega t - \beta_x x - \beta_y y - \beta_z z + \varphi_o\right)$$

$$= \mathrm{Re}\left\{\vec{E}_o e^{j\varphi_o} e^{j\omega t} e^{-j\beta_x x} e^{-j\beta_y y} e^{-j\beta_z z}\right\}$$

phasor $\Rightarrow \quad \vec{\mathrm{E}}(\vec{r}) = \vec{E}_o e^{j\varphi_o} e^{-j\beta_x x} e^{-j\beta_y y} e^{-j\beta_z z}$

$$\vec{H}(\vec{r},t) = \vec{H}_o \cos\left(\omega t - \beta_x x - \beta_y y - \beta_z z + \varphi_o\right)$$

$$= \mathrm{Re}\left\{\vec{H}_o e^{j\varphi_o} e^{j\omega t} e^{-j\beta_x x} e^{-j\beta_y y} e^{-j\beta_z z}\right\}$$

phasor $\Rightarrow \quad \vec{\mathrm{H}}(\vec{r}) = \vec{H}_o e^{j\varphi_o} e^{-j\beta_x x} e^{-j\beta_y y} e^{-j\beta_z z}$