We have obtained the following solutions for the steady-state voltage and current phasors in a transmission line:

Loss-less line

$$V(z) = V^{+} e^{-j\beta z} + V^{-} e^{j\beta z}$$

$$I(z) = \frac{1}{Z_0}\left(V^{+} e^{-j\beta z} - V^{-} e^{j\beta z}\right)$$

Lossy line

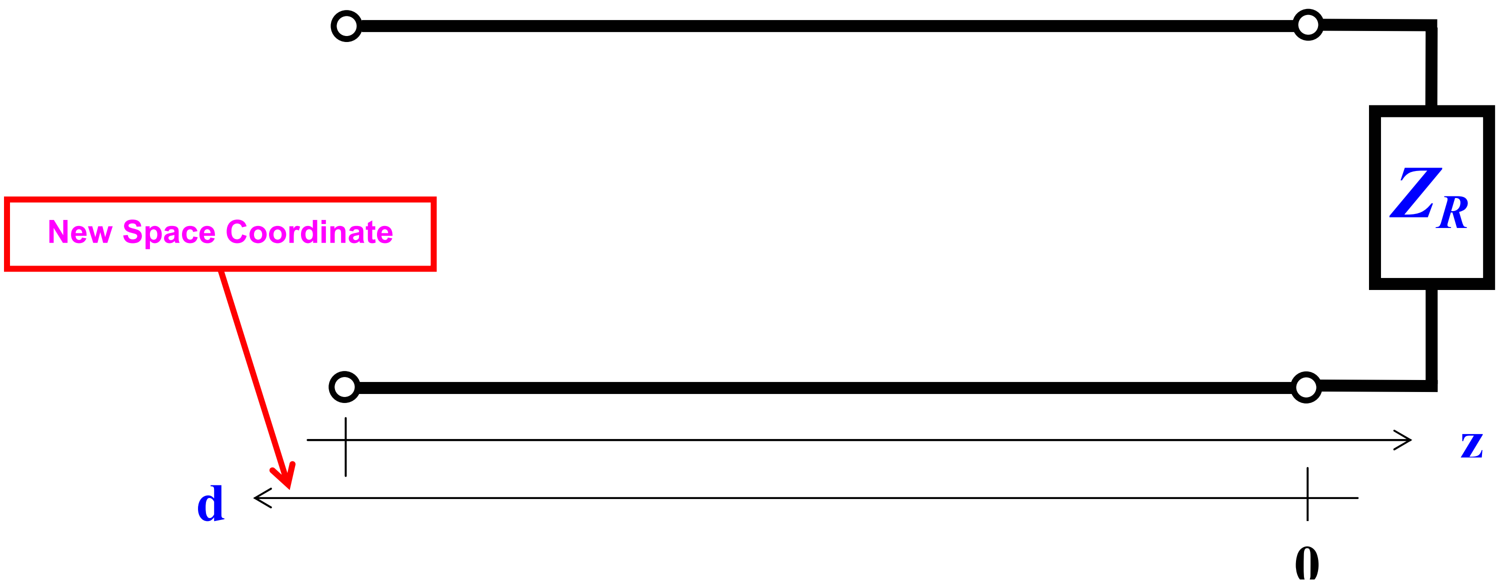
$$V(z) = V^{+} e^{-\gamma z} + V^{-} e^{\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left(V^{+} e^{-\gamma z} - V^{-} e^{\gamma z}\right)$$

Since $V(z)$ and $I(z)$ are the solutions of second order differential (wave) equations, we must determine two unknowns, V^{+} and V^{-}, which represent the amplitudes of steady-state voltage waves, travelling in the positive and in the negative direction, respectively.

Therefore, we need two boundary conditions to determine these unknowns, by considering the effect of the load and of the generator connected to the transmission line.

Before we consider the boundary conditions, it is very convenient to shift the reference of the space coordinate so that the zero reference is at the location of the load instead of the generator. Since the analysis of the transmission line normally starts from the load itself, this will simplify considerably the problem later.



We will also change the positive direction of the space coordinate, so that it increases when moving from load to generator along the transmission line.

We adopt a new coordinate d = – z, with zero reference at the load location. The new equations for voltage and current along the lossy transmission line are

Loss-less line

$$V(\mathrm{d}) = V^{+} e^{j\beta \mathrm{d}} + V^{-} e^{-j\beta \mathrm{d}}$$

$$I(\mathrm{d}) = \frac{1}{Z_0}\left(V^{+} e^{j\beta \mathrm{d}} - V^{-} e^{-j\beta \mathrm{d}}\right)$$

Lossy line

$$V(\mathrm{d}) = V^{+} e^{\gamma \mathrm{d}} + V^{-} e^{-\gamma \mathrm{d}}$$

$$I(\mathrm{d}) = \frac{1}{Z_0}\left(V^{+} e^{\gamma \mathrm{d}} - V^{-} e^{-\gamma \mathrm{d}}\right)$$

At the load (d = 0) we have, for both cases,

$$V(0) = V^{+} + V^{-}$$

$$I(0) = \frac{1}{Z_0}\left(V^{+} - V^{-}\right)$$

For a given load impedance Z_R, the load boundary condition is

$$V(0) = Z_R \, I(0)$$

Therefore, we have

$$V^{+} + V^{-} = \frac{Z_R}{Z_0}\left(V^{+} - V^{-}\right)$$

from which we obtain the voltage load reflection coefficient

$$\Gamma_R = \frac{V^{-}}{V^{+}} = \frac{Z_R - Z_0}{Z_R + Z_0}$$

We can introduce this result into the transmission line equations as

Loss-less line

$$V(\mathrm{d}) = V^+ e^{j\beta \mathrm{d}} \left(1 + \Gamma_R\, e^{-2j\beta \mathrm{d}}\right)$$

$$I(\mathrm{d}) = \frac{V^+ e^{j\beta \mathrm{d}}}{Z_0} \left(1 - \Gamma_R e^{-2j\beta \mathrm{d}}\right)$$

Lossy line

$$V(\mathrm{d}) = V^+ e^{\gamma \mathrm{d}} \left(1 + \Gamma_R\, e^{-2\gamma \mathrm{d}}\right)$$

$$I(\mathrm{d}) = \frac{V^+ e^{\gamma \mathrm{d}}}{Z_0} \left(1 - \Gamma_R e^{-2\gamma \mathrm{d}}\right)$$

At each line location we define a Generalized Reflection Coefficient

$$\Gamma(\mathrm{d}) = \Gamma_R\, e^{-2j\beta \mathrm{d}}$$

$$\Gamma(\mathrm{d}) = \Gamma_R\, e^{-2\gamma \mathrm{d}}$$

and the line equations become

$$V(\mathrm{d}) = V^+ e^{j\beta \mathrm{d}} \left(1 + \Gamma(\mathrm{d})\right)$$

$$I(\mathrm{d}) = \frac{V^+ e^{j\beta \mathrm{d}}}{Z_0} \left(1 - \Gamma(\mathrm{d})\right)$$

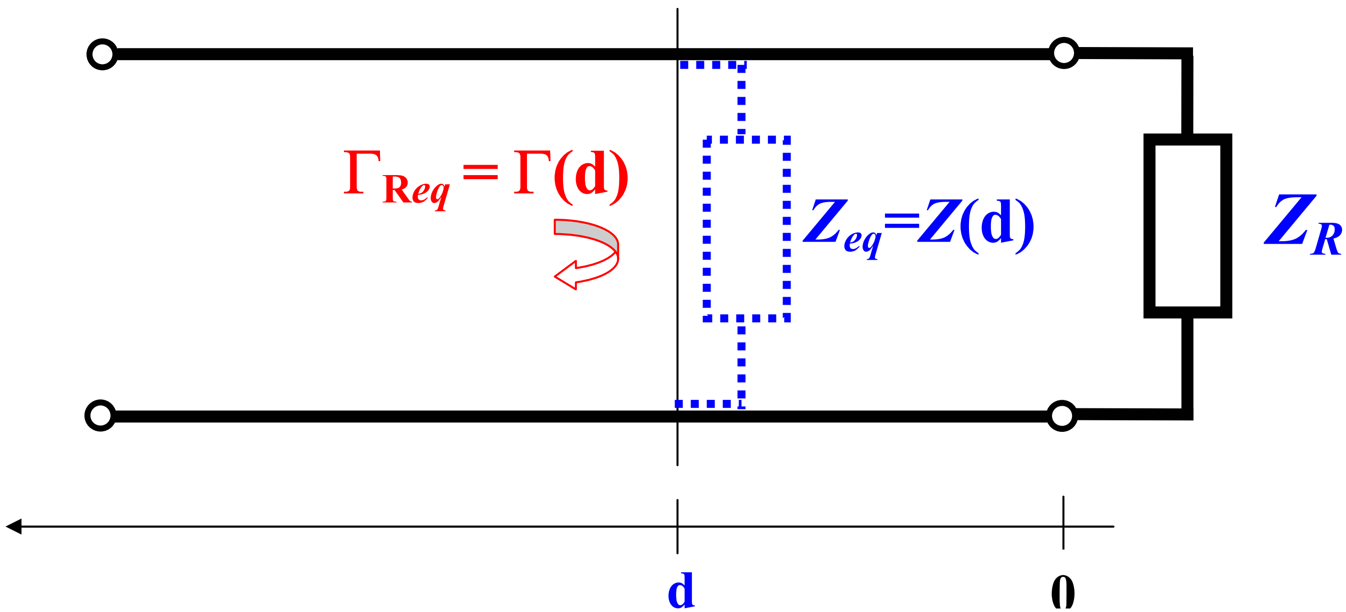
$$V(\mathrm{d}) = V^+ e^{\gamma \mathrm{d}} \left(1 + \Gamma(\mathrm{d})\right)$$

$$I(\mathrm{d}) = \frac{V^+ e^{\gamma \mathrm{d}}}{Z_0} \left(1 - \Gamma(\mathrm{d})\right)$$

We define the line impedance as

$$Z(\mathrm{d}) = \frac{V(\mathrm{d})}{I(\mathrm{d})} = Z_0 \frac{1+\Gamma(\mathrm{d})}{1-\Gamma(\mathrm{d})}$$

A simple circuit diagram can illustrate the significance of line impedance and generalized reflection coefficient:



If you imagine to cut the line at location d, the input impedance of the portion of line terminated by the load is the same as the line impedance at that location "before the cut". The behavior of the line on the left of location d is the same if an equivalent impedance with value *Z*(d) replaces the cut out portion. The reflection coefficient of the new load is equal to Γ(d)

$$\Gamma_{Req} = \Gamma(d) = \frac{Z_{Req} - Z_0}{Z_{Req} + Z_0}$$

If the total length of the line is L, the input impedance is obtained from the formula for the line impedance as

$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{V(\mathrm{L})}{I(\mathrm{L})} = Z_0 \frac{1+\Gamma(\mathrm{L})}{1-\Gamma(\mathrm{L})}$$

The input impedance is the equivalent impedance representing the entire line terminated by the load.

An important practical case is the low-loss transmission line, where the reactive elements still dominate but *R* and *G* cannot be neglected as in a loss-less line. We have the following conditions:

$$\omega L >> R \qquad\qquad \omega C >> G$$

so that

$$\begin{aligned}\gamma &= \sqrt{(j\omega L + R)(j\omega C + G)} \\ &= \sqrt{j\omega L\, j\omega C \left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\ &\approx j\omega\sqrt{LC}\sqrt{1 + \frac{R}{j\omega L} + \frac{G}{j\omega C} - \frac{RG}{\omega^2 LC}}\end{aligned}$$

The last term under the square root can be neglected, because it is the product of two very small quantities.

What remains of the square root can be expanded into a truncated Taylor series

$$\gamma \approx j\omega\sqrt{LC}\left[1+\frac{1}{2}\left(\frac{R}{j\omega L}+\frac{G}{j\omega C}\right)\right]$$

$$=\frac{1}{2}\left(R\sqrt{\frac{C}{L}}+G\sqrt{\frac{L}{C}}\right)+j\omega\sqrt{LC}$$

so that

$$\alpha=\frac{1}{2}\left(R\sqrt{\frac{C}{L}}+G\sqrt{\frac{L}{C}}\right) \qquad \beta=\omega\sqrt{LC}$$

The characteristic impedance of the low-loss line is a real quantity for all practical purposes and it is approximately the same as in a corresponding loss-less line

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{L}{C}}$$

and the phase velocity associated to the wave propagation is

$$v_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{LC}}$$

BUT NOTE:

In the case of the low-loss line, the equations for voltage and current retain the same form obtained for general lossy lines.

Again, we obtain the loss-less transmission line if we assume

$$R = 0 \qquad\qquad G = 0$$

This is often acceptable in relatively short transmission lines, where the overall attenuation is small.

As shown earlier, the characteristic impedance in a loss-less line is exactly real

$$Z_0 = \sqrt{\frac{L}{C}}$$

while the propagation constant has no attenuation term

$$\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} = j\beta$$

The loss-less line does not dissipate power, because α = 0.

For all cases, the line impedance was defined as

$$Z(\mathrm{d}) = \frac{V(\mathrm{d})}{I(\mathrm{d})} = Z_0 \frac{1+\Gamma(\mathrm{d})}{1-\Gamma(\mathrm{d})}$$

By including the appropriate generalized reflection coefficient, we can derive alternative expressions of the line impedance:

A) Loss-less line

$$Z(\mathrm{d}) = Z_0 \frac{1+\Gamma_R e^{-2j\beta \mathrm{d}}}{1-\Gamma_R e^{-2j\beta \mathrm{d}}} = Z_0 \frac{Z_R + jZ_0 \tan(\beta\, \mathrm{d})}{jZ_R \tan(\beta\, \mathrm{d}) + Z_0}$$

B) Lossy line (including low-loss)

$$Z(\mathrm{d}) = Z_0 \frac{1+\Gamma_R e^{-2\gamma \mathrm{d}}}{1-\Gamma_R e^{-2\gamma \mathrm{d}}} = Z_0 \frac{Z_R + Z_0 \tanh(\gamma\, \mathrm{d})}{Z_R \tanh(\gamma\, \mathrm{d}) + Z_0}$$

Let's now consider power flow in a transmission line, limiting the discussion to the time-average power, which accounts for the active power dissipated by the resistive elements in the circuit.

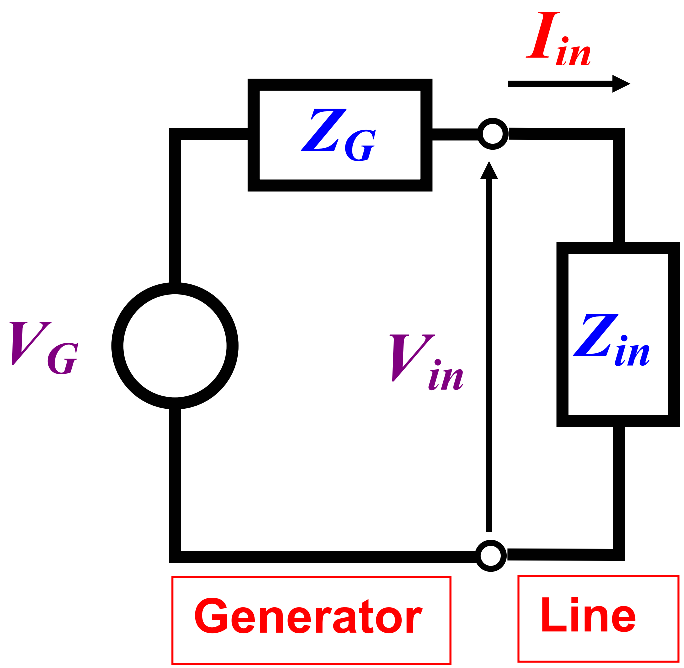
The time-average power at any transmission line location is

$$\langle P(\mathrm{d}, t) \rangle = \frac{1}{2} \mathrm{Re}\left\{ V(\mathrm{d})\, I^{*}(\mathrm{d}) \right\}$$

This quantity indicates the time-average power that flows through the line cross-section at location d. In other words, this is the power that, given a certain input, is able to reach location d and then flows into the remaining portion of the line beyond this point.

It is a common mistake to think that the quantity above is the power dissipated at location d !

The generator, the input impedance, the input voltage and the input current determine the power injected at the transmission line input.



$$V_{in} = V_G \frac{Z_{in}}{Z_G + Z_{in}}$$

$$I_{in} = V_G \frac{1}{Z_G + Z_{in}}$$

$$\langle P_{in} \rangle = \frac{1}{2} \mathrm{Re}\left\{ V_{in} \, I_{in}^{*} \right\}$$

The time-average power reaching the load of the transmission line is given by the general expression

$$\langle\, P(\mathrm{d}{=}0\,,t)\,\rangle = \frac{1}{2}\,\mathrm{Re}\left\{V(0)\,I^{*}(0)\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{V^{+}(1+\Gamma_R)\,\frac{1}{Z_0^{*}}\Big(V^{+}(1-\Gamma_R)\Big)^{*}\right\}$$

This represents the power dissipated by the load.

The time-average power absorbed by the line is simply the difference between the input power and the power absorbed by the load

$$\langle\, P_{line}\,\rangle = \langle\, P_{in}\,\rangle - \langle\, P(\mathrm{d}=0\,,t)\,\rangle$$

In a loss-less transmission line no power is absorbed by the line, so the input time-average power is the same as the time-average power absorbed by the load. Remember that the internal impedance of the generator dissipates part of the total power generated.

It is instructive to develop further the general expression for the time-average power at the load, using $Z_0=R_0+jX_0$ for the characteristic impedance, so that

$$\frac{1}{Z_0^*} = \frac{Z_0}{Z_0^* Z_0} = \frac{R_0 + jX_0}{|Z_0|^2} = \frac{R_0 + jX_0}{R_0^2 + X_0^2}$$

Alternatively, one may simplify the analysis by introducing the line characteristic admittance

$$Y_0 = \frac{1}{Z_0} = G_0 + jB_0$$

It may be more convenient to deal with the complex admittance at the numerator of the power expression, rather than the complex characteristic impedance at the denominator.

$$\langle P(\mathrm{d}=0,t)\rangle = \frac{1}{2}\mathrm{Re}\left\{ V^{+}\left(1+\Gamma_R\right)\frac{1}{Z_0^*}\left(V^{+}\left(1-\Gamma_R\right)\right)^* \right\} =$$

$$= \frac{\left|V^{+}\right|^2}{2\left|Z_0\right|^2}\mathrm{Re}\left\{ \left(R_0 + jX_0\right)\left(1+\mathrm{Re}\left\{\Gamma_R\right\}+j\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\left(1-\mathrm{Re}\left\{\Gamma_R\right\}+j\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\right\}$$

$$= \frac{\left|V^{+}\right|^2}{2\left|Z_0\right|^2}\mathrm{Re}\left\{ \left(R_0 + jX_0\right)\left(1-\left(\left(\mathrm{Re}\left\{\Gamma_R\right\}\right)^2+\left(\mathrm{Im}\left\{\Gamma_R\right\}\right)^2\right)+j2\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\right\}$$

$$= \frac{\left|V^{+}\right|^2}{2\left|Z_0\right|^2}\mathrm{Re}\left\{ \left(R_0 + jX_0\right)\left(1-\left|\Gamma_R\right|^2+j2\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\right\} =$$

$$= \frac{\left|V^{+}\right|^2}{2\left|Z_0\right|^2}\left[R_0 - R_0\left|\Gamma_R\right|^2 - 2X_0\,\mathrm{Im}\left\{\Gamma_R\right\}\right]$$

Equivalently, using the complex characteristic admittance:

$$\langle P(\mathrm{d}{=}0,t)\rangle = \frac{1}{2}\mathrm{Re}\left\{ V^+\left(1+\Gamma_R\right) Y_0^* \left(V^+\left(1-\Gamma_R\right)\right)^* \right\} =$$

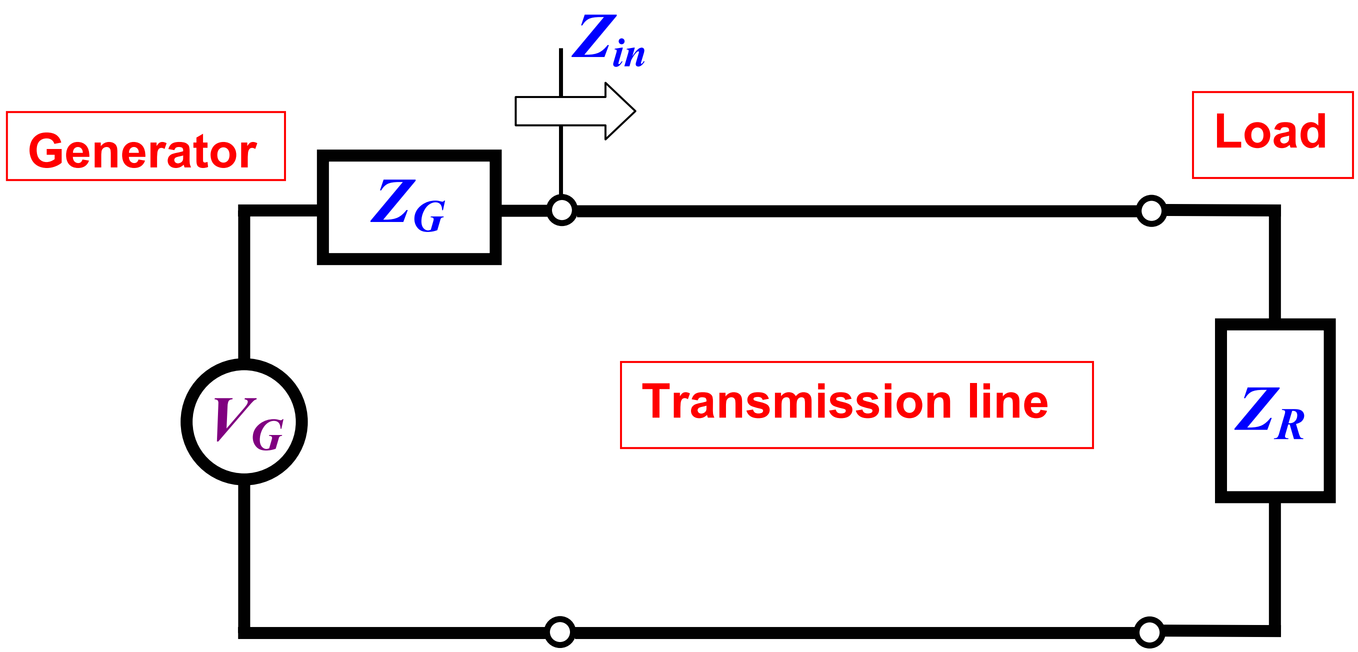
$$= \frac{\left|V^+\right|^2}{2}\mathrm{Re}\left\{ \left(G_0 - jB_0\right) \left(1+\mathrm{Re}\left\{\Gamma_R\right\} + j\,\mathrm{Im}\left\{\Gamma_R\right\}\right) \left(1-\mathrm{Re}\left\{\Gamma_R\right\} + j\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\right\}$$

$$= \frac{\left|V^+\right|^2}{2}\mathrm{Re}\left\{ \left(G_0 - jB_0\right) \left(1-\left(\left(\mathrm{Re}\left\{\Gamma_R\right\}\right)^2 + \left(\mathrm{Im}\left\{\Gamma_R\right\}\right)^2\right) + j2\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\right\}$$

$$= \frac{\left|V^+\right|^2}{2}\mathrm{Re}\left\{ \left(G_0 - jB_0\right) \left(1-\left|\Gamma_R\right|^2 + j2\,\mathrm{Im}\left\{\Gamma_R\right\}\right)\right\} =$$

$$= \frac{\left|V^+\right|^2}{2}\left[G_0 - G_0\left|\Gamma_R\right|^2 + 2B_0\,\mathrm{Im}\left\{\Gamma_R\right\}\right]$$

The time-average power, injected into the input of the transmission line, is maximized when the input impedance of the transmission line and the internal generator impedance are complex conjugate of each other.



$$Z_G = Z_{in}^{*}$$ **for maximum power transfer**

The characteristic impedance of the loss-less line is real and we can express the power flow, anywhere on the line, as

$$\langle P(\mathrm{d},t)\rangle = \frac{1}{2}\,\mathrm{Re}\{V(\mathrm{d})\,I^*(\mathrm{d})\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{V^+ e^{j\beta \mathrm{d}}\left(1+\Gamma_R e^{-j2\beta \mathrm{d}}\right)\right.$$

$$\left.\frac{1}{Z_0}(V^+)^* e^{-j\beta \mathrm{d}}\left(1-\Gamma_R e^{-j2\beta \mathrm{d}}\right)^*\right\}$$

$$= \underbrace{\frac{1}{2Z_0}\left|V^+\right|^2}_{\text{Incident wave}} - \underbrace{\frac{1}{2Z_0}\left|V^+\right|^2\left|\Gamma_R\right|^2}_{\text{Reflected wave}}$$

This result is valid for any location, including the input and the load, since the transmission line does not absorb any power.

In the case of low-loss lines, the characteristic impedance is again real, but the time-average power flow is position dependent because the line absorbs power.

$$\langle P(\mathrm{d},t)\rangle = \frac{1}{2}\,\mathrm{Re}\left\{ V(\mathrm{d})\, I^{*}(\mathrm{d}) \right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{ V^{+} e^{\alpha \mathrm{d}} e^{j\beta \mathrm{d}} \left(1+\Gamma_R e^{-2\gamma \mathrm{d}}\right) \frac{1}{Z_0} (V^{+})^{*} e^{\alpha \mathrm{d}} e^{-j\beta \mathrm{d}} \left(1-\Gamma_R e^{-2\gamma \mathrm{d}}\right)^{*} \right\}$$

$$= \underbrace{\frac{1}{2Z_0}\left|V^{+}\right|^{2} e^{2\alpha \mathrm{d}}}_{\text{Incident wave}} - \underbrace{\frac{1}{2Z_0}\left|V^{+}\right|^{2} e^{-2\alpha \mathrm{d}} \left|\Gamma_R\right|^{2}}_{\text{Reflected wave}}$$

Note that in a lossy line the reference for the amplitude of the incident voltage wave is at the load and that the amplitude grows exponentially moving towards the input. The amplitude of the incident wave behaves in the following way

$$\underbrace{V^+ e^{\alpha L}}_{\text{input}} \Leftrightarrow \underbrace{V^+ e^{\alpha d}}_{\text{inside the line}} \Leftrightarrow \underbrace{V^+}_{\text{load}}$$

The reflected voltage wave has maximum amplitude at the load, and it decays exponentially moving back towards the generator. The amplitude of the reflected wave behaves in the following way

$$\underbrace{V^+ \Gamma_R \, e^{-\alpha L}}_{\text{input}} \Leftrightarrow \underbrace{V^+ \Gamma_R \, e^{-\alpha d}}_{\text{inside the line}} \Leftrightarrow \underbrace{V^+ \Gamma_R}_{\text{load}}$$

For a general lossy line the power flow is again position dependent. Since the characteristic impedance is complex, the result has an additional term involving the imaginary part of the characteristic admittance, B_0, as

$$\langle P(\mathrm{d},t)\rangle = \frac{1}{2}\mathrm{Re}\left\{ V(\mathrm{d})\, I^*(\mathrm{d}) \right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{ V^+ e^{\alpha \mathrm{d}} e^{j\beta \mathrm{d}} \left(1+\Gamma(\mathrm{d})\right) \right.$$

$$\left. Y_0^* (V^+)^* e^{\alpha \mathrm{d}} e^{-j\beta \mathrm{d}} \left(1-\Gamma(\mathrm{d})\right)^* \right\}$$

$$= \frac{G_0}{2}\left|V^+\right|^2 \; e^{2\alpha \mathrm{d}} - \frac{G_0}{2}\left|V^+\right|^2 \; e^{-2\alpha \mathrm{d}} \left|\Gamma_R\right|^2$$

$$+ B_0 \left|V^+\right|^2 \; e^{2\alpha \mathrm{d}}\, \mathrm{Im}(\Gamma(\mathrm{d}))$$

For the general lossy line, keep in mind that

$$Y_0 = \frac{1}{Z_0} = \frac{Z_0^*}{Z_0\, Z_0^*} = \frac{R_0 - jX_0}{|Z_0|^2} = \frac{R_0 - jX_0}{R_0^2 + X_0^2} = G_0 + jB_0$$

$$G_0 = \frac{R_0}{R_0^2 + X_0^2} \qquad B_0 = \frac{-X_0}{R_0^2 + X_0^2}$$

Recall that for a low-loss transmission line the characteristic impedance is approximately real, so that

$$B_0 \approx 0 \quad \textbf{and} \quad Z_0 \approx 1/G_0 \approx R_0.$$

The previous result for the low-loss line can be readily recovered from the time-average power for the general lossy line.

To completely specify the transmission line problem, we still have to determine the value of $V+$ from the input boundary condition.

- **The load boundary condition imposes the <u>shape</u> of the interference pattern of voltage and current along the line.**
- **The input boundary condition, linked to the generator, imposes the <u>scaling</u> for the interference patterns.**

We have

$$V_{in} = V(\mathrm{L}) = V_G \frac{Z_{in}}{Z_G + Z_{in}} \quad \textbf{with} \quad Z_{in} = Z_0 \frac{1+\Gamma(\mathrm{L})}{1-\Gamma(\mathrm{L})}$$

or

$$\begin{cases} Z_{in} = Z_0 \dfrac{Z_R + jZ_0 \tan(\beta\,\mathrm{L})}{jZ_R \tan(\beta\,\mathrm{L}) + Z_0} & \textbf{loss - less line} \\[2ex] Z_{in} = Z_0 \dfrac{Z_R + Z_0 \tanh(\gamma\,\mathrm{L})}{Z_R \tanh(\gamma\,\mathrm{L}) + Z_0} & \textbf{lossy line} \end{cases}$$

For a loss-less transmission line:

$$V(\mathrm{L}) = V^{+} e^{j\beta\,\mathrm{L}} \left[1 + \Gamma(\mathrm{L})\right] = V^{+} e^{j\beta\,\mathrm{L}} (1 + \Gamma_R e^{-j2\beta\,\mathrm{L}})$$

$$\Rightarrow \qquad V^{+} = V_G \frac{Z_{in}}{Z_G + Z_{in}} \; \frac{1}{e^{j\beta\,\mathrm{L}} (1 + \Gamma_R e^{-j2\beta\,\mathrm{L}})}$$

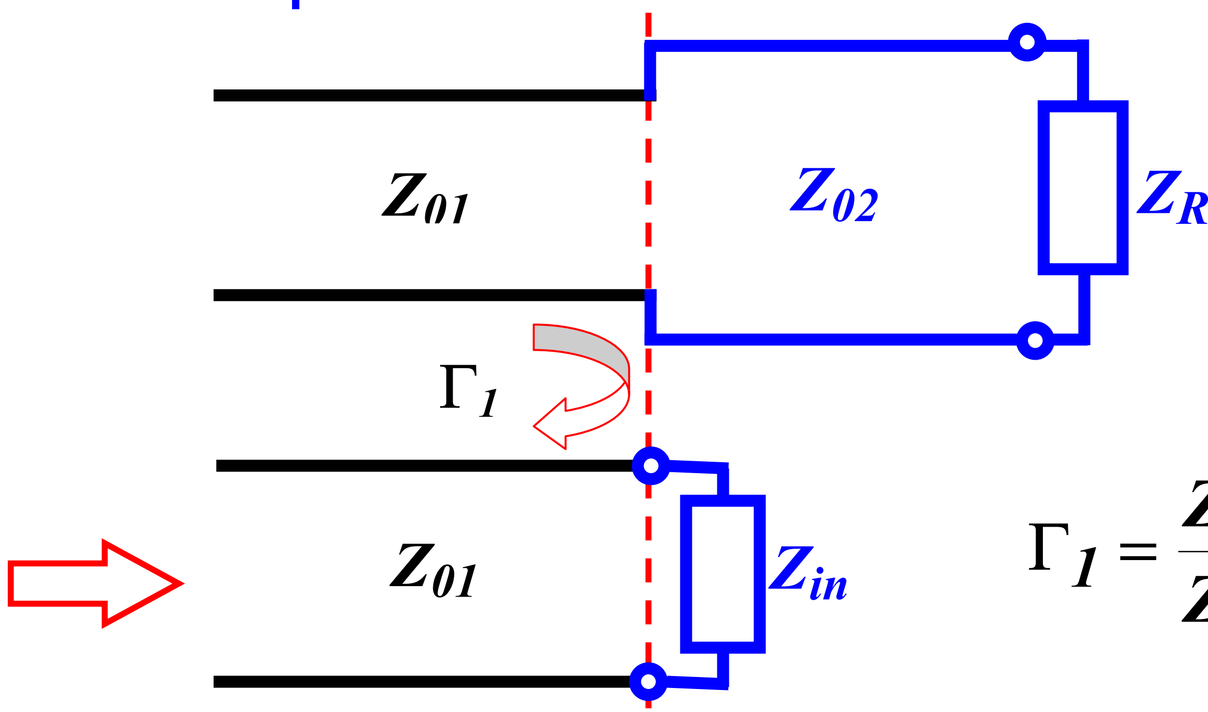
For a lossy transmission line:

$$V(\mathrm{L}) = V^{+} e^{\gamma\,\mathrm{L}} \left[1 + \Gamma(\mathrm{L})\right] = V^{+} e^{\gamma\,\mathrm{L}} \left(1 + \Gamma_R e^{-2\gamma\,\mathrm{d}}\right)$$

$$\Rightarrow \qquad V^{+} = V_G \frac{Z_{in}}{Z_G + Z_{in}} \; \frac{1}{e^{\gamma\,\mathrm{L}} (1 + \Gamma_R e^{-2\gamma\,\mathrm{L}})}$$

In order to have good control on the behavior of a high frequency circuit, it is very important to realize transmission lines as uniform as possible along their length, so that the impedance behavior of the line does not vary and can be easily characterized.

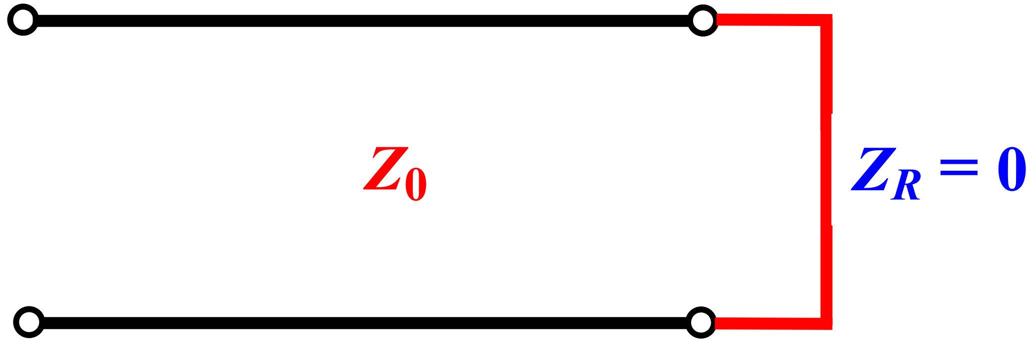
A change in transmission line properties, wanted or unwanted, entails a change in the characteristic impedance, which causes a reflection. Example:



$$\Gamma_1 = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}}$$

Special Cases

$Z_R \to 0$ (SHORT CIRCUIT)



The load boundary condition due to the short circuit is $V(0) = 0$

$$\Rightarrow V(d=0) = V^+ e^{j\beta 0}(1 + \Gamma_R e^{-j2\beta 0})$$

$$= V^+(1 + \Gamma_R) = 0$$

$$\Rightarrow \quad \Gamma_R = -1$$

Since

$$\Gamma_R = \frac{V^-}{V^+}$$

$$\Rightarrow \quad V^- = -V^+$$

We can write the line voltage phasor as

$$\begin{aligned} V(\mathrm{d}) &= V^+ e^{j\beta \mathrm{d}} + V^- e^{-j\beta \mathrm{d}} \\ &= V^+ e^{j\beta \mathrm{d}} - V^+ e^{-j\beta \mathrm{d}} \\ &= V^+ (e^{j\beta \mathrm{d}} - e^{-j\beta \mathrm{d}}) \\ &= 2\, j V^+ \sin(\beta \mathrm{d}) \end{aligned}$$

For the line current phasor we have

$$I(\mathrm{d}) = \frac{1}{Z_0}\left(V^+ e^{j\beta \mathrm{d}} - V^- e^{-j\beta \mathrm{d}}\right)$$

$$= \frac{1}{Z_0}\left(V^+ e^{j\beta \mathrm{d}} + V^+ e^{-j\beta \mathrm{d}}\right)$$

$$= \frac{V^+}{Z_0}\left(e^{j\beta \mathrm{d}} + e^{-j\beta \mathrm{d}}\right)$$

$$= \frac{2V^+}{Z_0}\cos(\beta \mathrm{d})$$

The line impedance is given by

$$Z(\mathrm{d}) = \frac{V(\mathrm{d})}{I(\mathrm{d})} = \frac{2\,jV^+ \sin(\beta \mathrm{d})}{2V^+ \cos(\beta \mathrm{d}) / Z_0} = jZ_0 \tan(\beta \mathrm{d})$$

The time-dependent values of voltage and current are obtained as

$$V(\mathrm{d},t) = \mathrm{Re}[V(\mathrm{d})\, e^{j\omega t}] = \mathrm{Re}[2\,j\,|V^+|\,e^{j\theta}\,\sin(\beta\,\mathrm{d})\, e^{j\omega t}]$$
$$= 2|V^+|\sin(\beta\,\mathrm{d})\cdot \mathrm{Re}[\,j\,e^{j(\omega t+\theta)}]$$
$$= 2|V^+|\sin(\beta\,\mathrm{d})\cdot \mathrm{Re}[\,j\cos(\omega t+\theta) - \sin(\omega t+\theta)]$$
$$= -2|V^+|\sin(\beta\,\mathrm{d})\sin(\omega t+\theta)$$

$$I(\mathrm{d},t) = \mathrm{Re}[I(\mathrm{d})\, e^{j\omega t}] = \mathrm{Re}[2\,|V^+|\,e^{j\theta}\cos(\beta\,\mathrm{d})\, e^{j\omega t}]\,/\,Z_0$$
$$= 2|V^+|\cos(\beta\,\mathrm{d})\cdot \mathrm{Re}[\,e^{j(\omega t+\theta)}]\,/\,Z_0$$
$$= 2|V^+|\cos(\beta\,\mathrm{d})\cdot \mathrm{Re}[\,(\cos(\omega t+\theta) + j\sin(\omega t+\theta)]\,/\,Z_0$$
$$= 2\frac{|V^+|}{Z_0}\cos(\beta\,\mathrm{d})\cos(\omega t+\theta)$$

The time-dependent power is given by

$$P(\mathrm{d},t) = V(\mathrm{d},t)\cdot I(\mathrm{d},t)$$

$$= -4\frac{|V^+|^2}{Z_0}\sin(\beta\,\mathrm{d})\cos(\beta\,\mathrm{d})\sin(\omega t+\theta)\cos(\omega t+\theta)$$

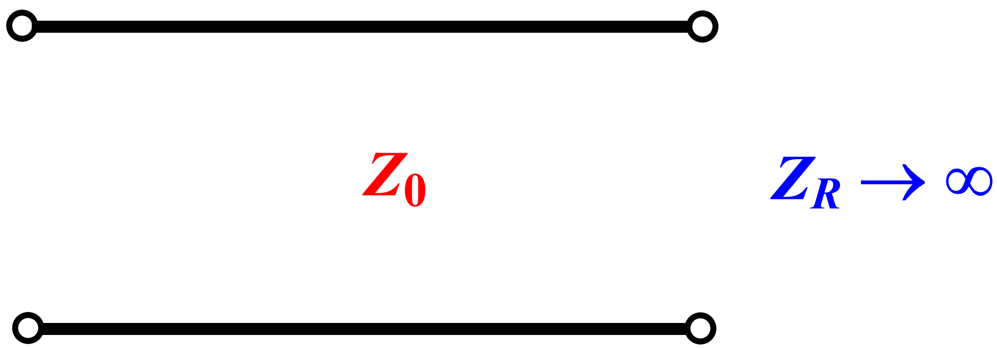
$$= -\frac{|V^+|^2}{Z_0}\sin(2\beta\,\mathrm{d})\sin\,(2\omega t+2\theta)$$

and the corresponding time-average power is

$$< P(\mathrm{d},t) > \; = \frac{1}{T}\int_0^T P(\mathrm{d},t)\,dt$$

$$= -\frac{|V^+|^2}{Z_0}\sin(2\beta\,\mathrm{d})\frac{1}{T}\int_0^T \sin\,(2\omega t+2\theta) = 0$$

$Z_R \to \infty$ (OPEN CIRCUIT)



The load boundary condition due to the open circuit is $I(0) = 0$

$$\Rightarrow I(d=0) = \frac{V^+}{Z_0} e^{j\beta 0} \left(1 - \Gamma_R \, e^{-j2\beta 0}\right)$$

$$= \frac{V^+}{Z_0} (1 - \Gamma_R) = 0$$

$$\Rightarrow \quad \Gamma_R = 1$$

Since

$$\Gamma_R = \frac{V^-}{V^+}$$

$$\Rightarrow \quad V^- = V^+$$

We can write the line current phasor as

$$I(\mathrm{d}) = \frac{1}{Z_0}(V^+ e^{j\beta \mathrm{d}} - V^- e^{-j\beta \mathrm{d}})$$

$$= \frac{1}{Z_0}(V^+ e^{j\beta \mathrm{d}} - V^+ e^{-j\beta \mathrm{d}})$$

$$= \frac{V^+}{Z_0}(e^{j\beta \mathrm{d}} - e^{-j\beta \mathrm{d}}) = \frac{2jV^+}{Z_0}\sin(\beta \mathrm{d})$$

For the line voltage phasor we have

$$
\begin{aligned}
V(\mathrm{d}) &= (V^+ e^{j\beta\,\mathrm{d}} + V^- e^{-j\beta\,\mathrm{d}}) \\
&= (V^+ e^{j\beta\,\mathrm{d}} + V^+ e^{-j\beta\,\mathrm{d}}) \\
&= V^+ (e^{j\beta\,\mathrm{d}} + e^{-j\beta\,\mathrm{d}}) \\
&= 2V^+ \cos(\beta\,\mathrm{d})
\end{aligned}
$$

The line impedance is given by

$$
Z(\mathrm{d}) = \frac{V(\mathrm{d})}{I(\mathrm{d})} = \frac{2V^+ \cos(\beta\,\mathrm{d})}{2\,jV^+ \sin(\beta\,\mathrm{d}) / Z_0} = -j\,\frac{Z_0}{\tan(\beta\,\mathrm{d})}
$$

The time-dependent values of voltage and current are obtained as

$$\begin{aligned}V(\mathrm{d},t) &= \mathrm{Re}[V(\mathrm{d})\,e^{j\omega t}] = \mathrm{Re}[2\,|V^+|\,e^{j\theta}\cos(\beta\,\mathrm{d})\,e^{j\omega t}]\\ &= 2|V^+|\cos(\beta\,\mathrm{d})\cdot\mathrm{Re}[\,e^{j(\omega t+\theta)}]\\ &= 2|V^+|\cos(\beta\,\mathrm{d})\cdot\mathrm{Re}[(\cos(\omega t+\theta)+j\sin(\omega t+\theta)]\\ &= 2|V^+|\cos(\beta\,\mathrm{d})\cos(\omega t+\theta)\end{aligned}$$

$$\begin{aligned}I(\mathrm{d},t) &= \mathrm{Re}[I(\mathrm{d})\,e^{j\omega t}] = \mathrm{Re}[2\,j\,|V^+|\,e^{j\theta}\sin(\beta\,\mathrm{d})\,e^{j\omega t}]/Z_0\\ &= 2|V^+|\sin(\beta\,\mathrm{d})\cdot\mathrm{Re}[\,j\,e^{j(\omega t+\theta)}]/Z_0\\ &= 2|V^+|\sin(\beta\,\mathrm{d})\cdot\mathrm{Re}[j\cos(\omega t+\theta)-\sin(\omega t+\theta)]/Z_0\\ &= -2\frac{|V^+|}{Z_0}\sin(\beta\,\mathrm{d})\sin(\omega t+\theta)\end{aligned}$$

The time-dependent power is given by

$$P(\mathrm{d},t) = V(\mathrm{d},t)\cdot I(\mathrm{d},t) =$$

$$= -4\frac{|V^+|^2}{Z_0}\cos(\beta\,\mathrm{d})\sin(\beta\,\mathrm{d})\cos(\omega t+\theta)\sin(\omega t+\theta)$$

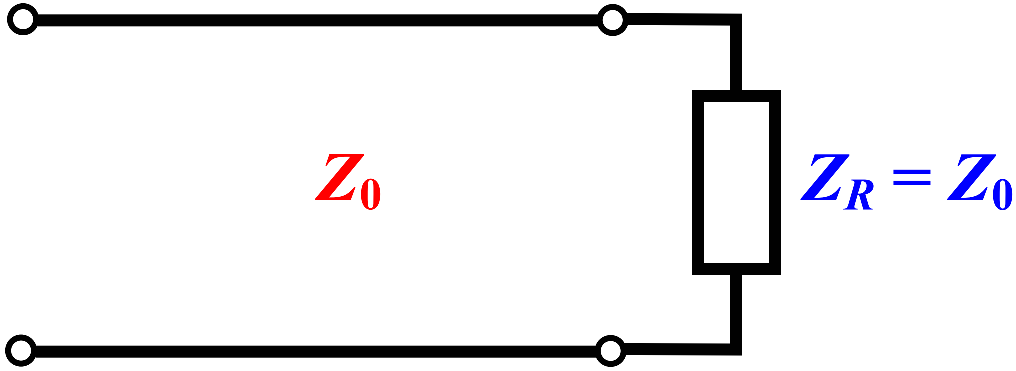
$$= -\frac{|V^+|^2}{Z_0}\sin(2\beta\,\mathrm{d})\sin\,(2\omega t+2\theta)$$

and the corresponding time-average power is

$$< P(\mathrm{d},t) > \; = \; \frac{1}{T}\int_0^T P(\mathrm{d},t)\,dt$$

$$= -\frac{|V^+|^2}{Z_0}\sin(2\beta\,\mathrm{d})\frac{1}{T}\int_0^T \sin\,(2\omega t+2\theta) = 0$$

$Z_R = Z_0$ (MATCHED LOAD)



The reflection coefficient for a matched load is

$$\Gamma_R = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

no reflection!

The line voltage and line current phasors are

$$V(\mathrm{d}) = V^+ e^{j\beta \mathrm{d}}(1 + \Gamma_R\, e^{-2j\beta \mathrm{d}}) = V^+ e^{j\beta \mathrm{d}}$$

$$I(d) = \frac{V^+}{Z_0} e^{j\beta \mathrm{d}}(1 - \Gamma_R\, e^{-2j\beta \mathrm{d}}) = \frac{V^+}{Z_0} e^{j\beta \mathrm{d}}$$

The line impedance is independent of position and equal to the characteristic impedance of the line

$$Z(\mathrm{d}) = \frac{V(\mathrm{d})}{I(\mathrm{d})} = \frac{V^{+} e^{j\beta \mathrm{d}}}{\dfrac{V^{+}}{Z_0} e^{j\beta \mathrm{d}}} = Z_0$$

The time-dependent voltage and current are

$$V(\mathrm{d},t) = \mathrm{Re}[\,|V^{+}|\, e^{j\theta}\, e^{j\beta \mathrm{d}}\, e^{j\omega t}]$$

$$= |V^{+}| \cdot \mathrm{Re}[e^{j(\omega t + \beta \mathrm{d} + \theta)}] = |V^{+}| \cos(\omega t + \beta \mathrm{d} + \theta)$$

$$I(\mathrm{d},t) = \mathrm{Re}[\,|V^{+}|\, e^{j\theta}\, e^{j\beta \mathrm{d}}\, e^{j\omega t}] / Z_0$$

$$= \frac{|V^{+}|}{Z_0} \cdot \mathrm{Re}[e^{j(\omega t + \beta \mathrm{d} + \theta)}] = \frac{|V^{+}|}{Z_0} \cos(\omega t + \beta \mathrm{d} + \theta)$$

The time-dependent power is

$$P(\mathrm{d},t) = |V^+| \cos(\omega t + \beta \mathrm{d} + \theta) \frac{|V^+|}{Z_0} \cos(\omega t + \beta \mathrm{d} + \theta)$$

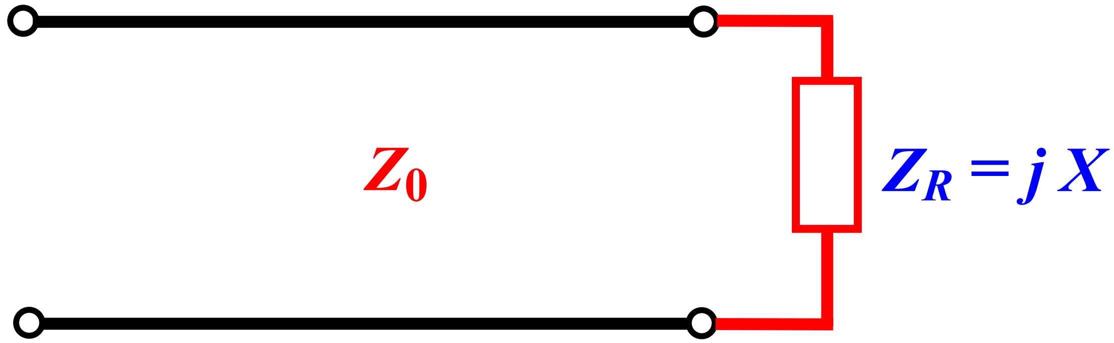
$$= \frac{|V^+|^2}{Z_0} \cos^2(\omega t + \beta \mathrm{d} + \theta)$$

and the time average power absorbed by the load is

$$< P(\mathrm{d}) > = \frac{1}{T} \int_0^t \frac{|V^+|^2}{Z_0} \cos^2(\omega t + \beta \mathrm{d} + \theta)\, dt$$

$$= \frac{|V^+|^2}{2 Z_0}$$

$Z_R = jX$ (PURE REACTANCE)



The reflection coefficient for a purely reactive load is

$$\Gamma_R = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{jX - Z_0}{jX + Z_0} =$$

$$= \frac{(jX - Z_0)(jX - Z_0)}{(jX + Z_0)(jX - Z_0)} = \frac{X^2 - Z_0^2}{Z_0^2 + X^2} + 2\,j\,\frac{XZ_0}{Z_0^2 + X^2}$$

In polar form

$$\Gamma_R = |\Gamma_R| \exp(j\theta)$$

where

$$|\Gamma_R| = \sqrt{\frac{\left(X^2 - Z_0^2\right)^2}{\left(Z_0^2 + X^2\right)^2} + \frac{4X^2 Z_0^2}{\left(Z_0^2 + X^2\right)^2}} = \sqrt{\frac{\left(Z_0^2 + X^2\right)^2}{\left(Z_0^2 + X^2\right)^2}} = 1$$

$$\theta = \tan^{-1}\left(\frac{2XZ_0}{X^2 - Z_0^2}\right)$$

The reflection coefficient has unitary magnitude, as in the case of short and open circuit load, with zero time average power absorbed by the load. Both voltage and current are finite at the load, and the time-dependent power oscillates between positive and negative values. This means that the load periodically stores power and then returns it to the line without dissipation.

Reactive impedances can be realized with transmission lines terminated by a short or by an open circuit. The input impedance of a loss-less transmission line of length L terminated by a short circuit is purely imaginary

$$Z_{in} = j Z_0 \tan(\beta \mathrm{L}) = j Z_0 \tan\left(\frac{2\pi}{\lambda}\mathrm{L}\right) = j Z_0 \tan\left(\frac{2\pi f}{v_p}\mathrm{L}\right)$$

For a specified frequency f, any reactance value (positive or negative!) can be obtained by changing the length of the line from 0 to $\lambda/2$. An inductance is realized for $\mathrm{L} < \lambda/4$ (positive tangent) while a capacitance is realized for $\lambda/4 < \mathrm{L} < \lambda/2$ (negative tangent).

When $\mathrm{L} = 0$ and $\mathrm{L} = \lambda/2$ the tangent is zero, and the input impedance corresponds to a short circuit. However, when $\mathrm{L} = \lambda/4$ the tangent is infinite and the input impedance corresponds to an open circuit.

Since the tangent function is periodic, the same impedance behavior of the impedance will repeat identically for each additional line increment of length $\lambda/2$. A similar periodic behavior is also obtained when the length of the line is fixed and the frequency of operation is changed.

At zero frequency (infinite wavelength), the short circuited line behaves as a short circuit for any line length. When the frequency is increased, the wavelength shortens and one obtains an inductance for $L < \lambda/4$ and a capacitance for $\lambda/4 < L < \lambda/2$, with an open circuit at $L = \lambda/4$ and a short circuit again at $L = \lambda/2$.

Note that the frequency behavior of lumped elements is very different. Consider an ideal inductor with inductance L assumed to be constant with frequency, for simplicity. At zero frequency the inductor also behaves as a short circuit, but the reactance varies monotonically and linearly with frequency as

$$X = \omega L \quad \textbf{(always an inductance)}$$

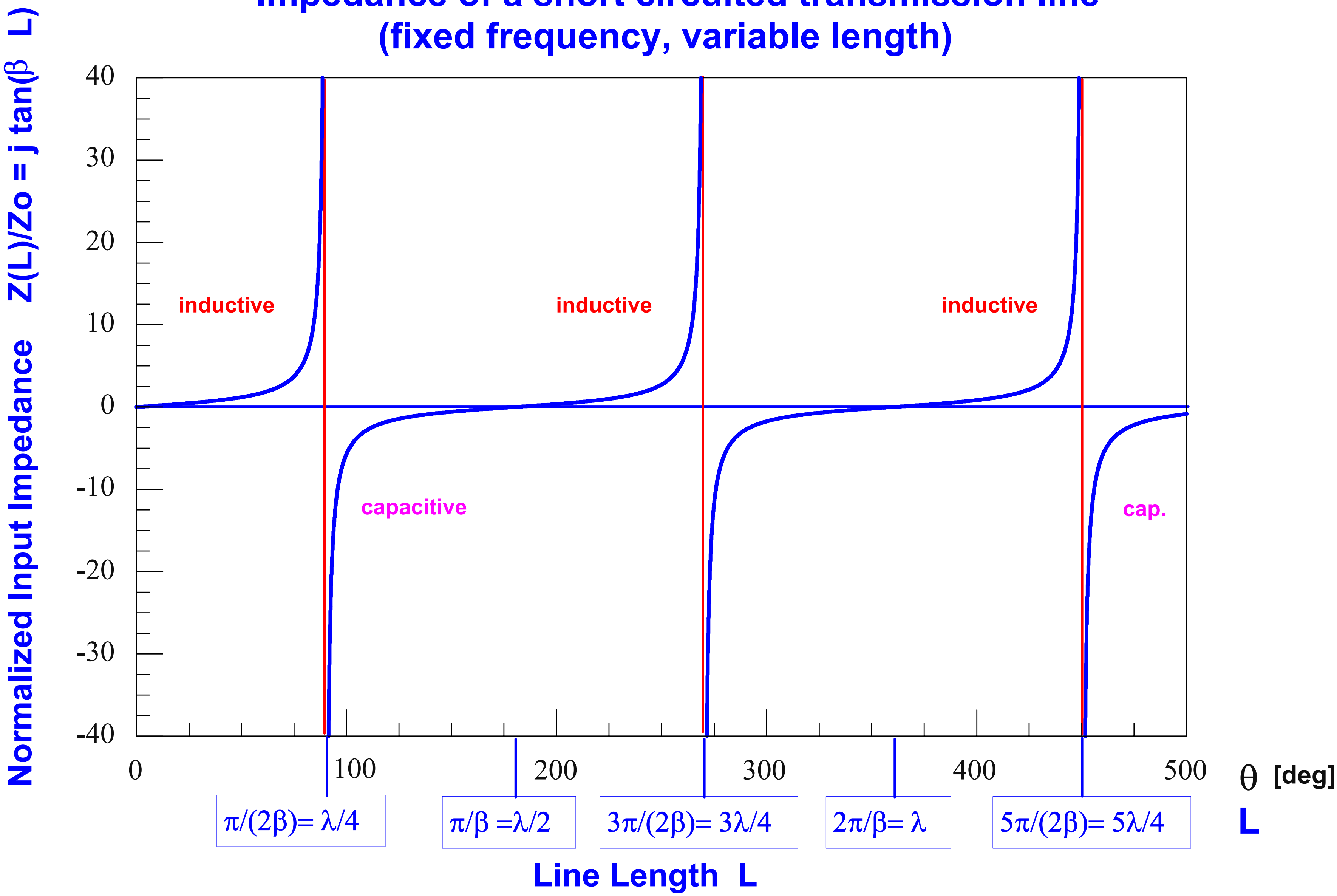
Short circuited transmission line – Fixed frequency

L ↓

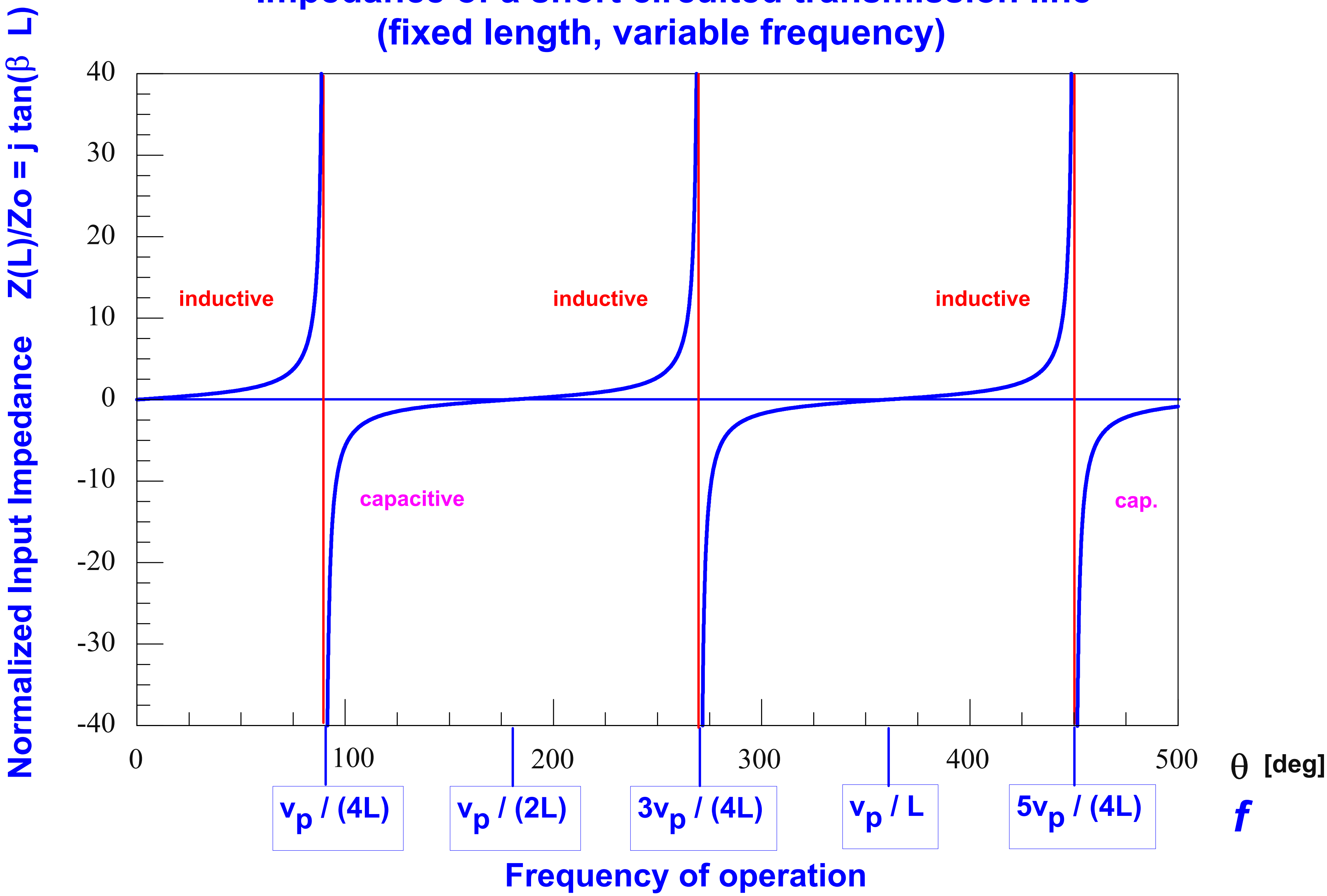
$L = 0$	$Z_{in} = 0$	short circuit
$0 < L < \frac{\lambda}{4}$	$\text{Im}\{Z_{in}\} > 0$	inductance
$L = \frac{\lambda}{4}$	$Z_{in} \to \infty$	open circuit
$\frac{\lambda}{4} < L < \frac{\lambda}{2}$	$\text{Im}\{Z_{in}\} < 0$	capacitance
$L = \frac{\lambda}{2}$	$Z_{in} = 0$	short circuit
$\frac{\lambda}{2} < L < \frac{3\lambda}{4}$	$\text{Im}\{Z_{in}\} > 0$	inductance
$L = \frac{3\lambda}{4}$	$Z_{in} \to \infty$	open circuit
$\frac{3\lambda}{4} < L < \lambda$	$\text{Im}\{Z_{in}\} < 0$	capacitance

…

Impedance of a short circuited transmission line (fixed frequency, variable length)



Impedance of a short circuited transmission line (fixed length, variable frequency)



For a transmission line of length $\mathbf{L}$ terminated by an open circuit, the input impedance is again purely imaginary

$$Z_{in} = -j\frac{Z_0}{\tan(\beta L)} = -j\frac{Z_0}{\tan\left(\frac{2\pi}{\lambda}L\right)} = -j\frac{Z_0}{\tan\left(\frac{2\pi f}{v_p}L\right)}$$

We can also use the open circuited line to realize any reactance, but starting from a capacitive value when the line length is very short.

Note once again that the frequency behavior of a corresponding lumped element is different. Consider an ideal capacitor with capacitance C assumed to be constant with frequency. At zero frequency the capacitor behaves as an open circuit, but the reactance varies monotonically and linearly with frequency as

$$X = \frac{1}{\omega C}$$ **(always a capacitance)**

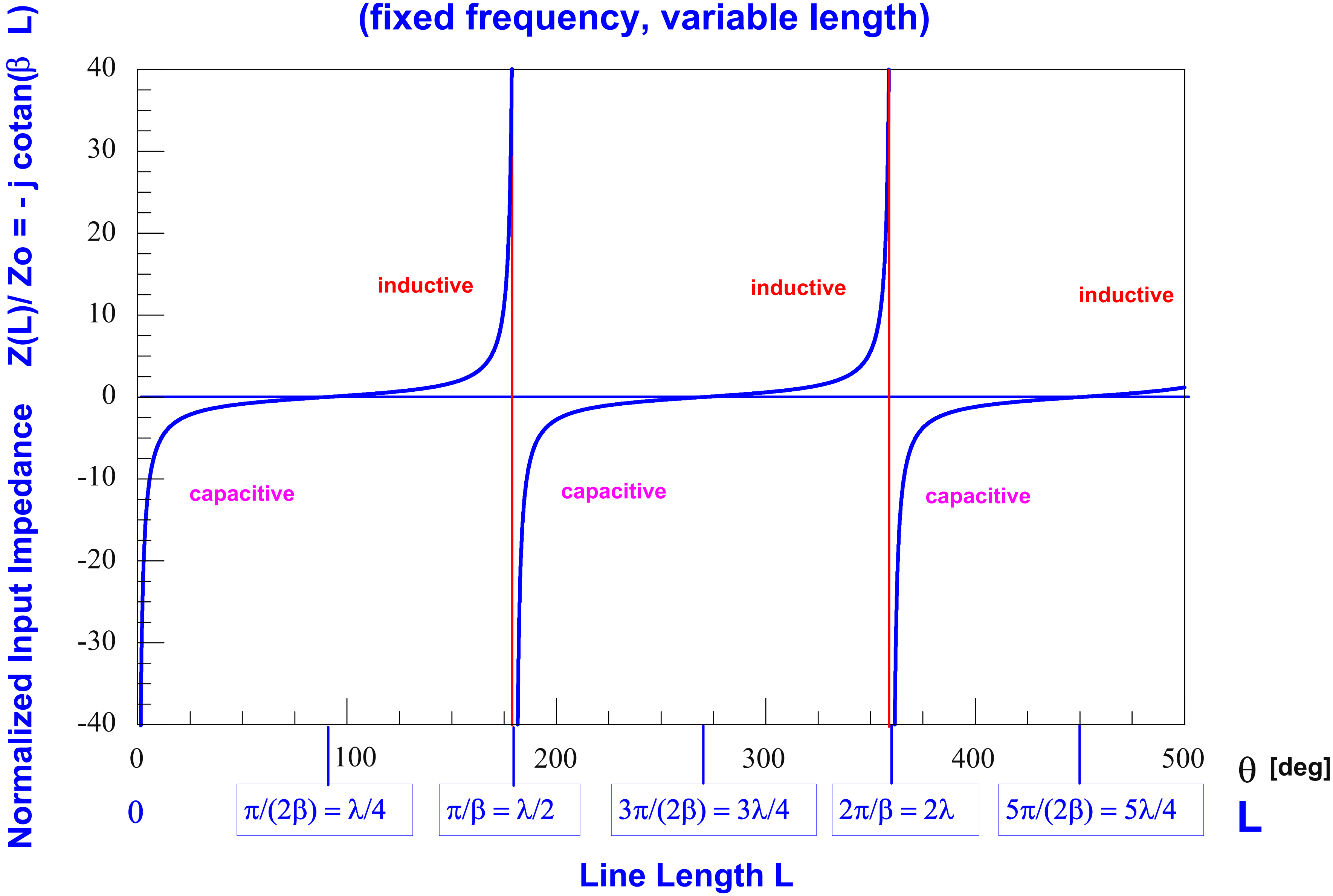
Open circuit transmission line – Fixed frequency

L ↓

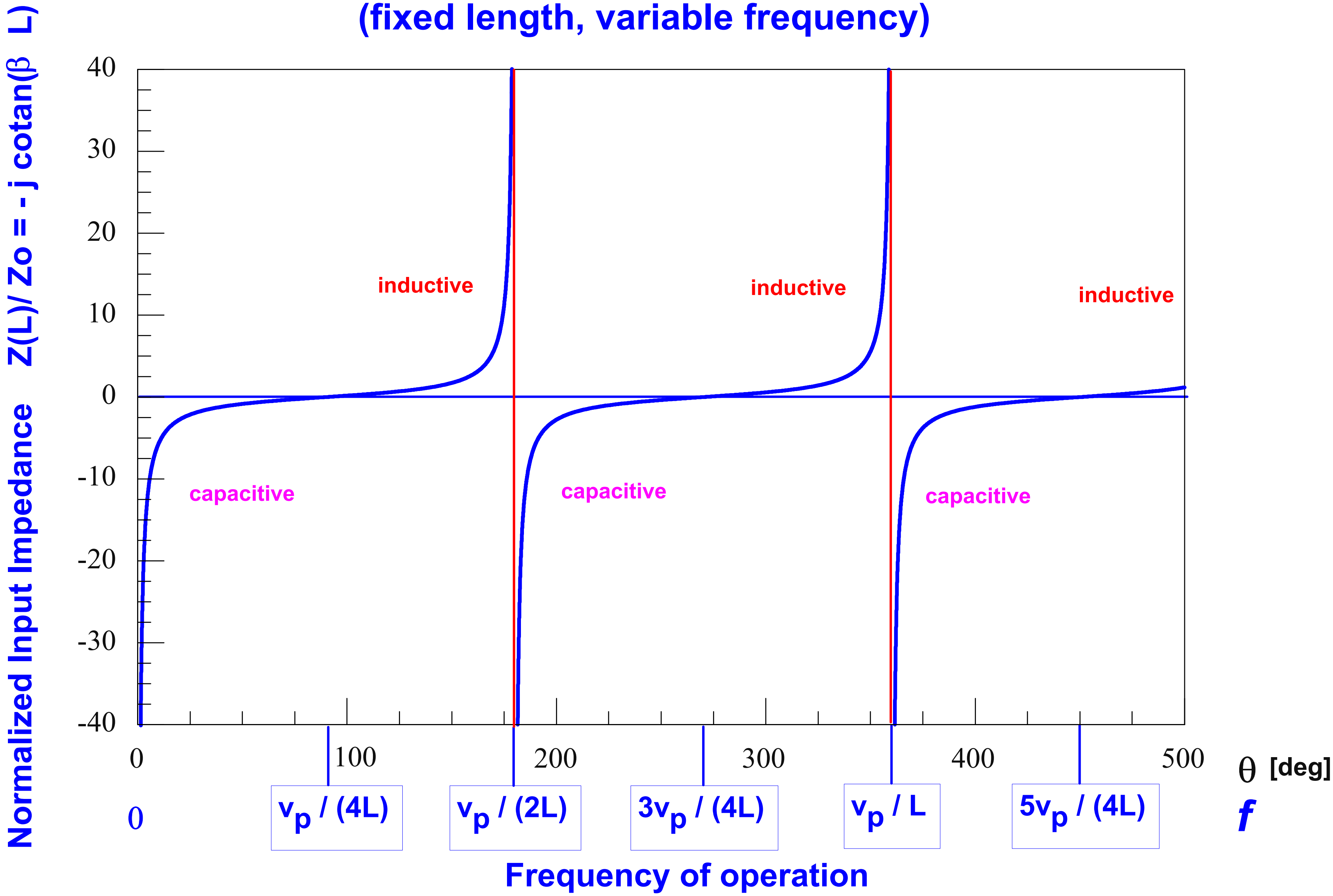
$L = 0$	$Z_{in} \to \infty$	open circuit
$0 < L < \frac{\lambda}{4}$	$\mathrm{Im}\{Z_{in}\} < 0$	capacitance
$L = \frac{\lambda}{4}$	$Z_{in} = 0$	short circuit
$\frac{\lambda}{4} < L < \frac{\lambda}{2}$	$\mathrm{Im}\{Z_{in}\} > 0$	inductance
$L = \frac{\lambda}{2}$	$Z_{in} \to \infty$	open circuit
$\frac{\lambda}{2} < L < \frac{3\lambda}{4}$	$\mathrm{Im}\{Z_{in}\} < 0$	capacitance
$L = \frac{3\lambda}{4}$	$Z_{in} = 0$	short circuit
$\frac{3\lambda}{4} < L < \lambda$	$\mathrm{Im}\{Z_{in}\} > 0$	inductance

…

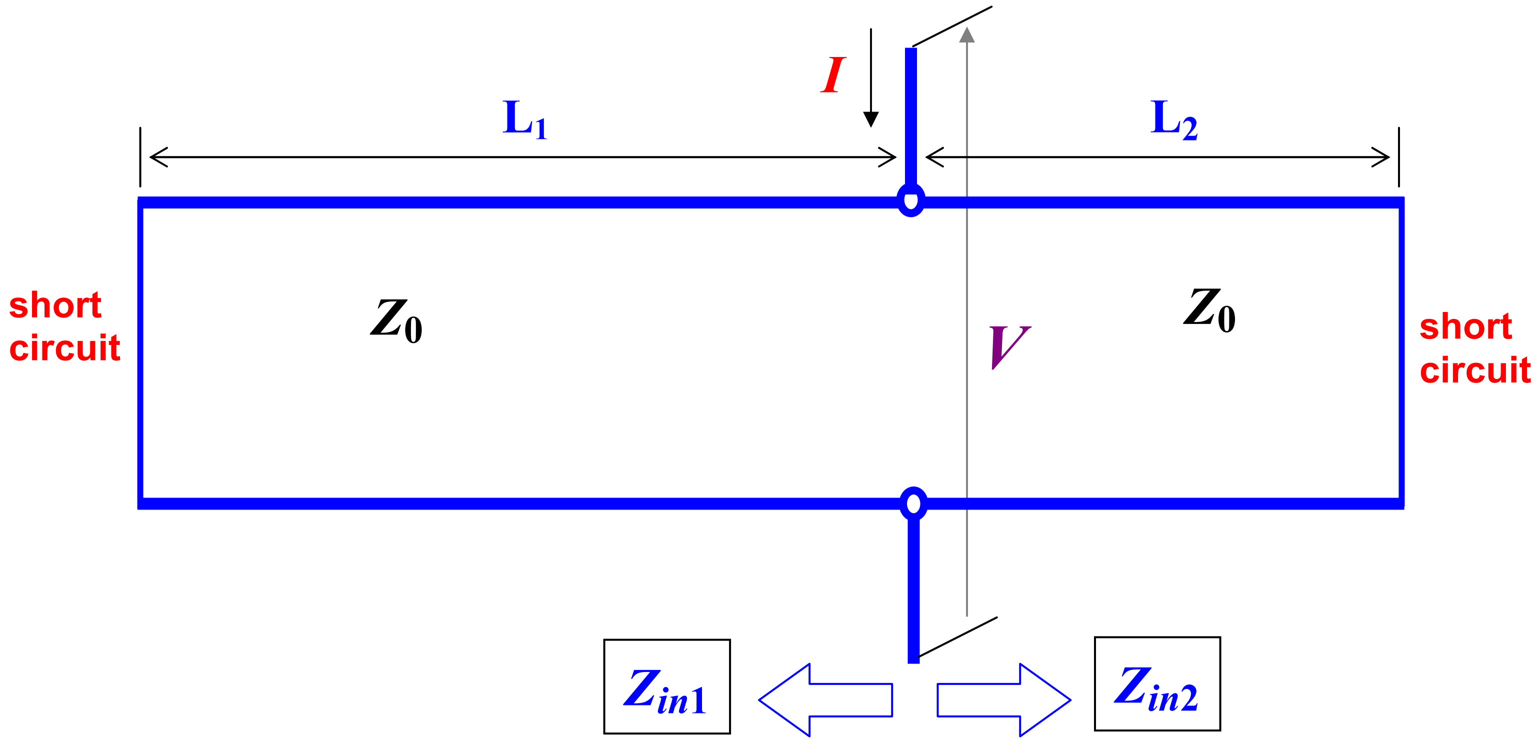
Impedance of an open circuited transmission line
(fixed frequency, variable length)



Impedance of an open circuited transmission line (fixed length, variable frequency)



It is possible to realize resonant circuits by using transmission lines as reactive elements. For instance, consider the circuit below realized with lines having the same characteristic impedance:



$$Z_{in1} = j Z_0 \tan\left(\beta L_1\right) \qquad Z_{in2} = j Z_0 \tan\left(\beta L_2\right)$$

The circuit is resonant if L_1 and L_2 are chosen such that an inductance and a capacitance are realized.

A resonance condition is established when the total input impedance of the parallel circuit is infinite or, equivalently, when the input admittance of the parallel circuit is zero

$$\frac{1}{jZ_0\tan\left(\beta_r L_1\right)}+\frac{1}{jZ_0\tan\left(\beta_r L_2\right)}=0$$

or

$$\tan\left(\frac{\omega_r}{v_p}L_1\right)=-\tan\left(\frac{\omega_r}{v_p}L_2\right) \qquad \textbf{with} \qquad \beta_r=\frac{2\pi}{\lambda_r}=\frac{\omega_r}{v_p}$$

Since the tangent is a periodic function, there is a multiplicity of possible resonant angular frequencies ω_r that satisfy the condition above. The values can be found by using a numerical procedure to solve the trascendental equation above.