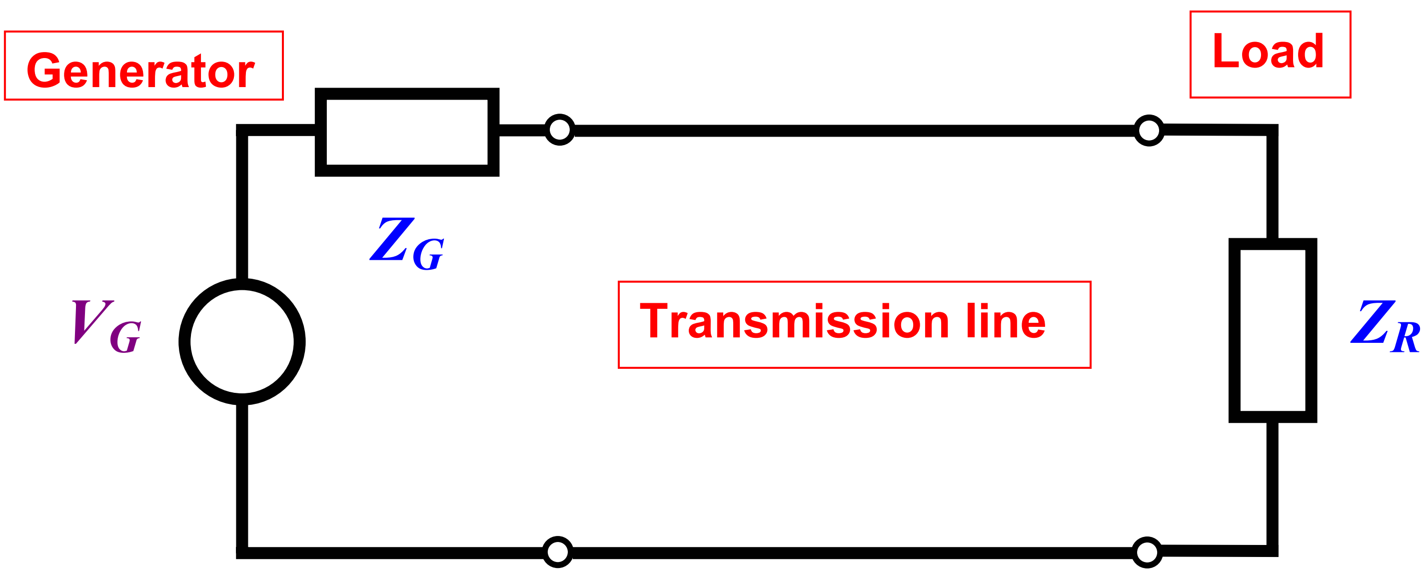
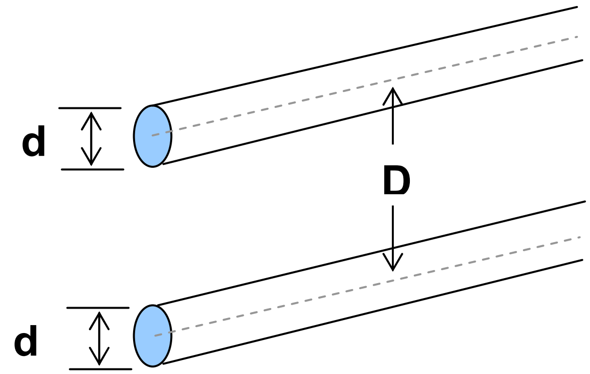
Transmission Line Equations

A typical engineering problem involves the transmission of a signal from a generator to a load. A transmission line is the part of the circuit that provides the direct link between generator and load.

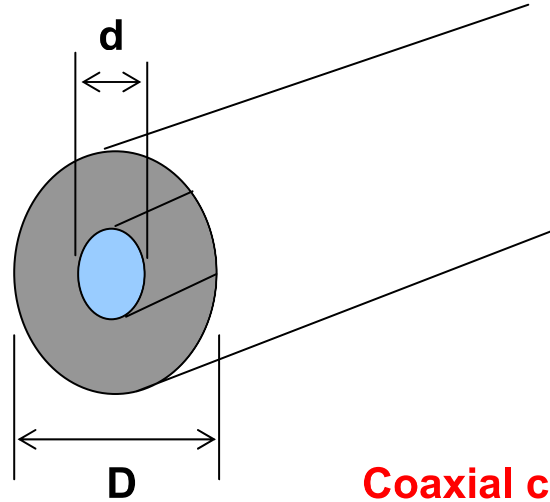
Transmission lines can be realized in a number of ways. Common examples are the parallel-wire line and the coaxial cable. For simplicity, we use in most diagrams the parallel-wire line to represent circuit connections, but the theory applies to all types of transmission lines.



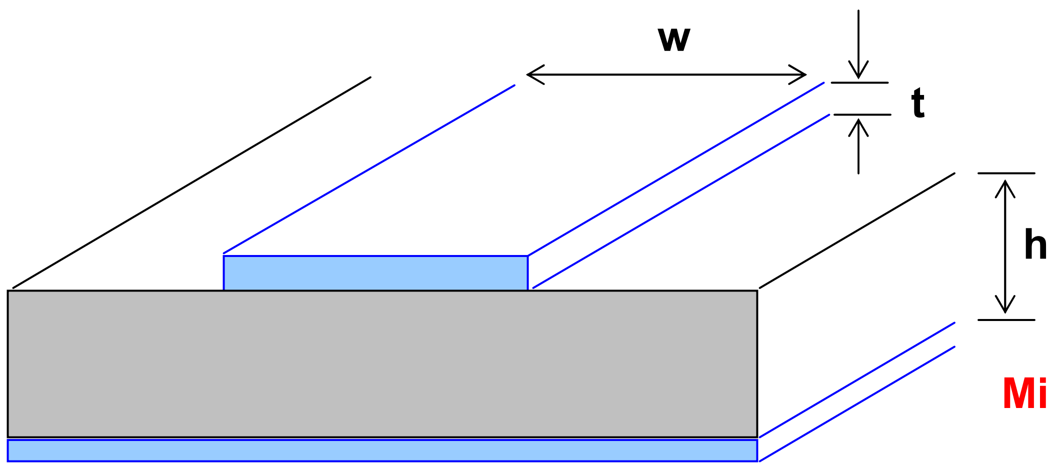
Examples of transmission lines



Two-wire line

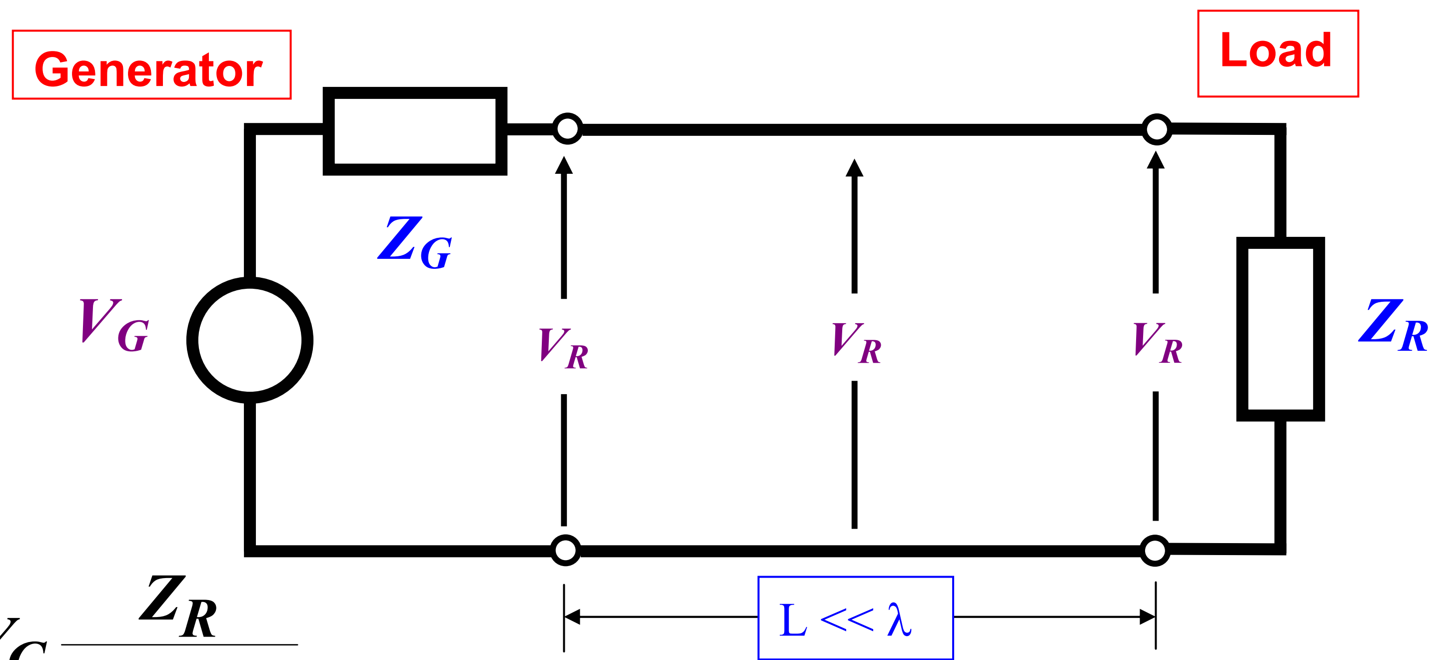


Coaxial cable



Microstrip

If you are only familiar with low frequency circuits, you are used to treat all lines connecting the various circuit elements as perfect wires, with no voltage drop and no impedance associated to them (lumped impedance circuits). This is a reasonable procedure as long as the length of the wires is much smaller than the wavelength of the signal. At any given time, the measured voltage and current are the same for each location on the same wire.



$$V_R = V_G \frac{Z_R}{Z_G + Z_R}$$

Let's look at some examples. The electricity supplied to households consists of high power sinusoidal signals, with frequency of 60Hz or 50Hz, depending on the country. Assuming that the insulator between wires is air ($\varepsilon \approx \varepsilon_0$), the wavelength for 60Hz is:

$$\lambda = \frac{c}{f} = \frac{2.999 \times 10^8}{60} \approx 5.0 \times 10^6 \; m = 5,000 \; km$$

which is the about the distance between S. Francisco and Boston!

Let's compare to a frequency in the microwave range, for instance 60 GHz. The wavelength is given by

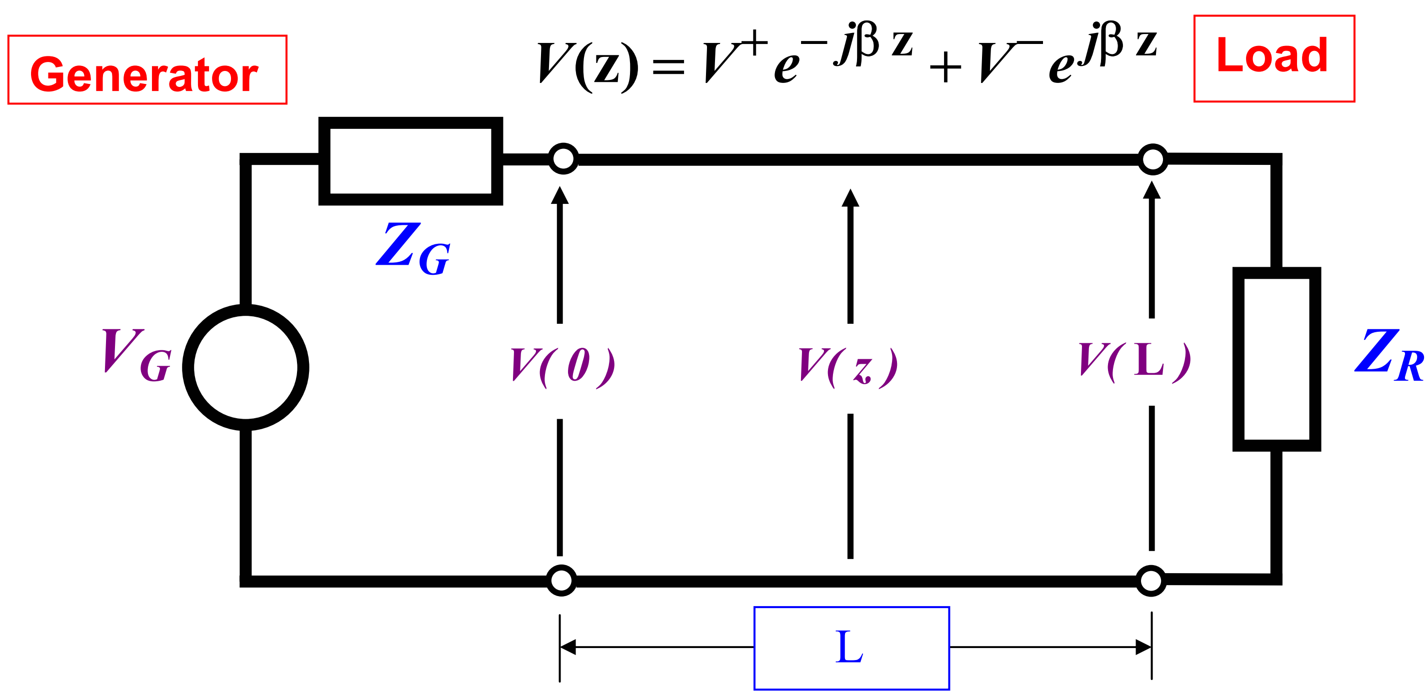
$$\lambda = \frac{c}{f} = \frac{2.999 \times 10^8}{60 \times 10^9} \approx 5.0 \times 10^{-3} \; m = 5.0 \; mm$$

which is comparable to the size of a microprocessor chip.

Which conclusions do you draw?

For sufficiently high frequencies the wavelength is comparable with the length of conductors in a transmission line. The signal propagates as a wave of voltage and current along the line, because it cannot change instantaneously at all locations. Therefore, we cannot neglect the impedance properties of the wires (distributed impedance circuits).

$$V(z) = V^{+} e^{-j\beta z} + V^{-} e^{j\beta z}$$



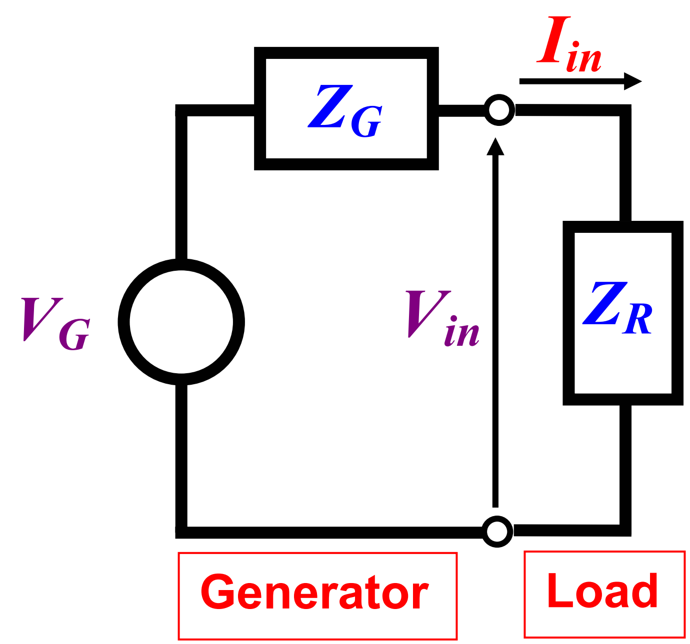
Note that the equivalent circuit of a generator consists of an ideal alternating voltage generator in series with its actual internal impedance. When the generator is open ($Z_R \to \infty$) we have:

$$I_{in} = 0 \quad \text{and} \quad V_{in} = V_G$$

If the generator is connected to a load Z_R

$$I_{in} = \frac{V_G}{(Z_G + Z_R)}$$

$$V_{in} = \frac{V_G\, Z_R}{(Z_G + Z_R)}$$



If the load is a short ($Z_R = 0$)

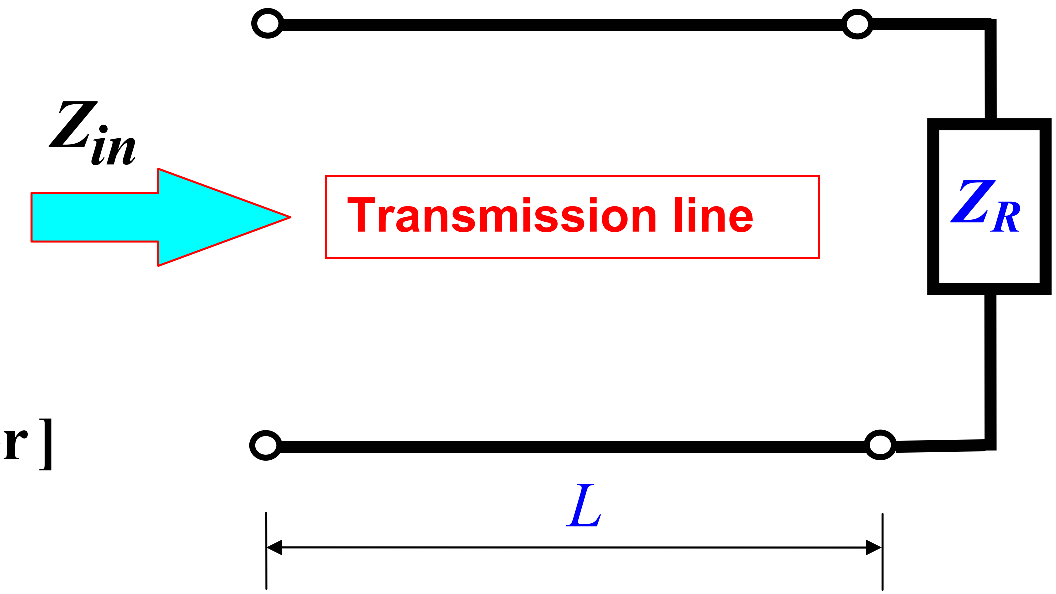
$$I_{in} = \frac{V_G}{Z_G} \quad \text{and} \quad V_{in} = 0$$

The simplest circuit problem that we can study consists of a voltage generator connected to a load through a uniform transmission line. In general, the impedance seen by the generator is not the same as the impedance of the load, because of the presence of the transmission line, except for some very particular cases:

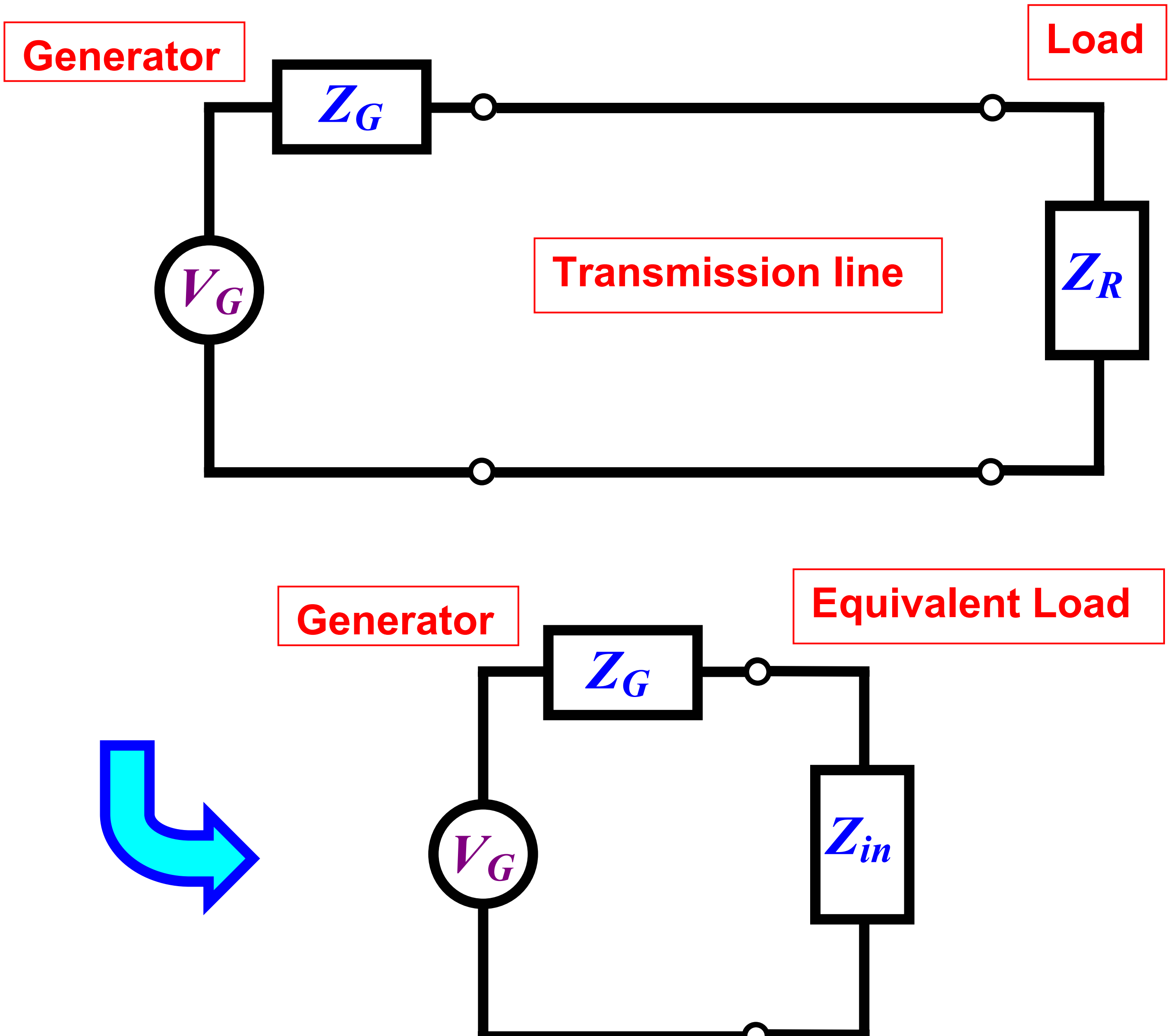
$$Z_{in} = Z_R$$

only if

$$L = n\frac{\lambda}{2} \qquad [\, n = \text{integer} \,]$$

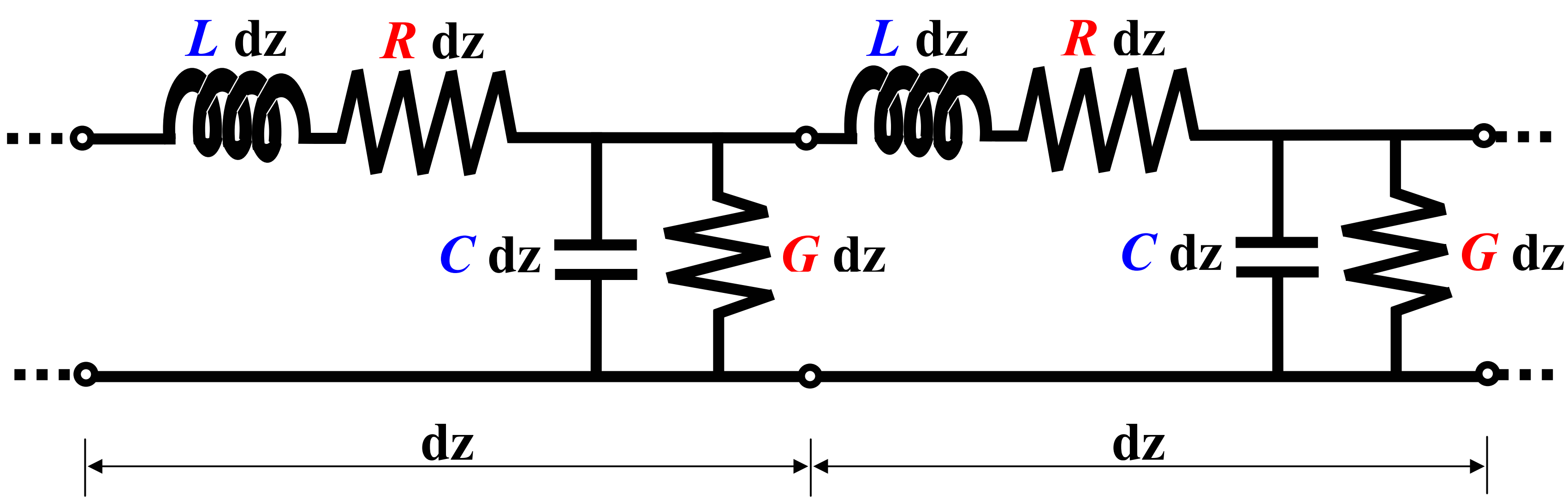


Our first goal is to determine the equivalent impedance seen by the generator, that is, the input impedance of a line terminated by the load. Once that is known, standard circuit theory can be used.



Generator
Z_G
Load
V_G
Transmission line
Z_R
Generator
Z_G
Equivalent Load
V_G
Z_{in}

A uniform transmission line is a "distributed circuit" that we can describe as a cascade of identical cells with infinitesimal length. The conductors used to realize the line possess a certain series inductance and resistance. In addition, there is a shunt capacitance between the conductors, and even a shunt conductance if the medium insulating the wires is not perfect. We use the concept of shunt conductance, rather than resistance, because it is more convenient for adding the parallel elements of the shunt. We can represent the uniform transmission line with the distributed circuit below (general lossy line)



The impedance parameters L, R, C, and G represent:

L = series inductance per unit length
R = series resistance per unit length
C = shunt capacitance per unit length
G = shunt conductance per unit length.

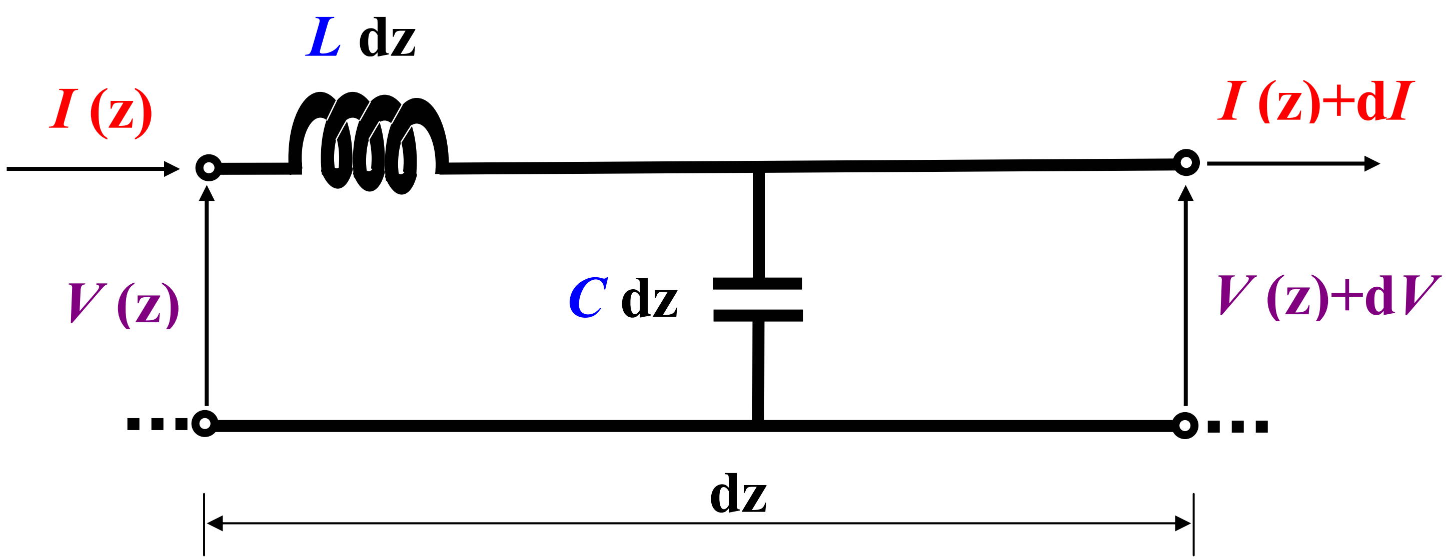
Each cell of the distributed circuit will have impedance elements with values: $L\mathrm{dz}$, $R\mathrm{dz}$, $C\mathrm{dz}$, and $G\mathrm{dz}$, where dz is the infinitesimal length of the cells.

If we can determine the differential behavior of an elementary cell of the distributed circuit, in terms of voltage and current, we can find a global differential equation that describes the entire transmission line. We can do so, because we assume the line to be uniform along its length.

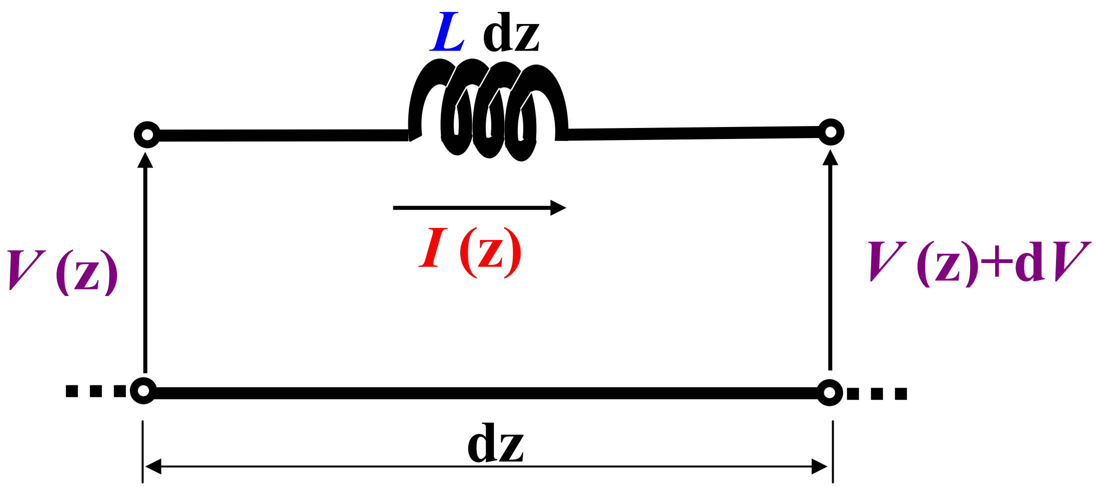
So, all we need to do is to study how voltage and current vary in a single elementary cell of the distributed circuit.

❑ Loss-less Transmission Line

In many cases, it is possible to neglect resistive effects in the line. In this approximation there is no Joule effect loss because only reactive elements are present. The equivalent circuit for the elementary cell of a loss-less transmission line is shown in the figure below.



The series inductance determines the variation of the voltage from input to output of the cell, according to the sub-circuit below



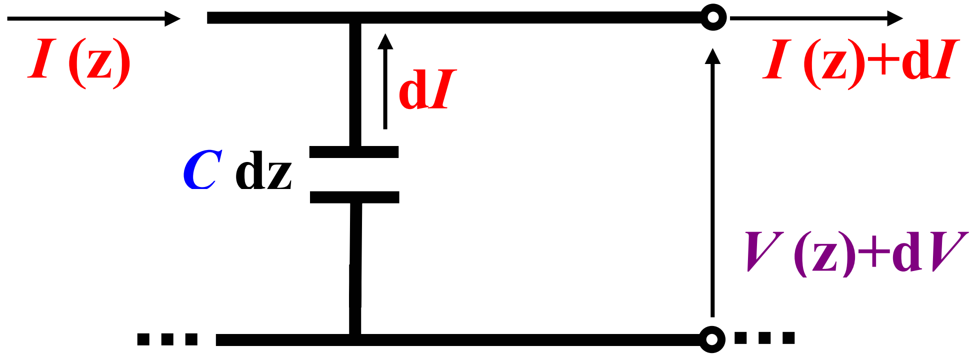
The corresponding circuit equation is

$$(V + \mathrm{d}V) - V = -\, j\omega L \,\mathrm{d}z\, I$$

which gives a first order differential equation for the voltage

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -\, j\omega L\, I$$

The current flowing through the shunt capacitance determines the variation of the current from input to output of the cell.



The circuit equation for the sub-circuit above is

$$\mathrm{d}I = -\,j\omega C \mathrm{d}z\,(V + \mathrm{d}V) = -\,j\omega C V \mathrm{d}z - j\omega C\,\mathrm{d}V\,\mathrm{d}z$$

The second term (including $\mathrm{d}V\,\mathrm{d}z$) tends to zero very rapidly in the limit of infinitesimal length $\mathrm{d}z$ leaving a first order differential equation for the current

$$\frac{\mathrm{d}I}{\mathrm{d}z} = -\,j\omega C\,V$$

We have obtained a system of two coupled first order differential equations that describe the behavior of voltage and current on the uniform loss-less transmission line. The equations must be solved simultaneously.

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{dz}} = -\,j\omega L\; I \\ \dfrac{\mathrm{d}I}{\mathrm{dz}} = -\,j\omega C\, V \end{cases}$$

These are often called "telegraphers' equations" of the loss-less transmission line.

One can easily obtain a set of uncoupled equations by differentiating with respect to the space coordinate. The first order differential terms are eliminated by using the corresponding telegraphers' equation

$$\boxed{\frac{\mathrm{d}I}{\mathrm{dz}} = -j\omega C\,V}$$

$$\frac{\mathrm{d}^2 V}{\mathrm{dz}^2} = -j\omega L\,\frac{\mathrm{d}I}{\mathrm{dz}} = j\omega L\; j\omega C V = -\omega^2\,LC\,V$$

$$\frac{\mathrm{d}^2 I}{\mathrm{dz}^2} = -j\omega C\,\frac{\mathrm{d}V}{\mathrm{dz}} = j\omega C\; j\omega L\,I = -\omega^2\,LC\,I$$

$$\boxed{\frac{\mathrm{d}V}{\mathrm{dz}} = -j\omega L\,I}$$

These are often called "telephonists' equations".

We have now two uncoupled second order differential equations for voltage and current, which give an equivalent description of the loss-less transmission line. Mathematically, these are wave equations and can be solved independently.

The general solution for the voltage equation is

$$V(z) = V^{+} e^{-j\beta z} + V^{-} e^{j\beta z}$$

where the wave propagation constant is

$$\beta = \omega\sqrt{LC}$$

Note that the complex exponential terms including β have unitary magnitude and purely "imaginary" argument, therefore they only affect the "phase" of the wave in space.

We have the following useful relations:

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{v_p} = \frac{\omega}{v_p}$$

$$= \frac{\omega\sqrt{\varepsilon_r \mu_r}}{c} = \omega\sqrt{\varepsilon_0 \mu_0}\sqrt{\varepsilon_r \mu_r} = \omega\sqrt{\varepsilon \mu}$$

Here, $\lambda = v_p / f$ is the wavelength of the dielectric medium surrounding the conductors of the transmission line and

$$v_p = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}} = \frac{1}{\sqrt{\varepsilon \mu}}$$

is the phase velocity of an electromagnetic wave in the dielectric.

As you can see, the propagation constant β can be written in many different, equivalent ways.

The current distribution on the transmission line can be readily obtained by differentiation of the result for the voltage

$$\frac{\mathrm{d}V}{\mathrm{dz}} = -j\beta V^{+} e^{-j\beta z} + j\beta V^{-} e^{j\beta z} = -j\omega L\ I$$

which gives

$$I(z) = \sqrt{\frac{C}{L}}\left(V^{+} e^{-j\beta z} - V^{-} e^{j\beta z}\right) = \frac{1}{Z_0}\left(V^{+} e^{-j\beta z} - V^{-} e^{j\beta z}\right)$$

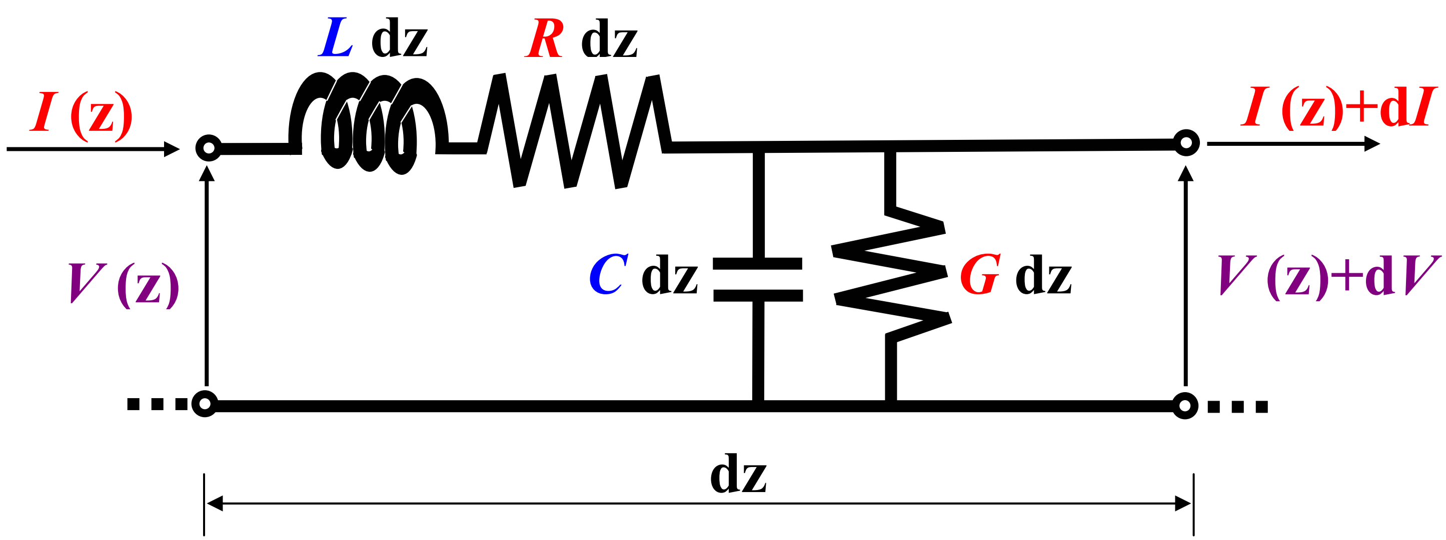
The real quantity

$$Z_0 = \sqrt{\frac{L}{C}}$$

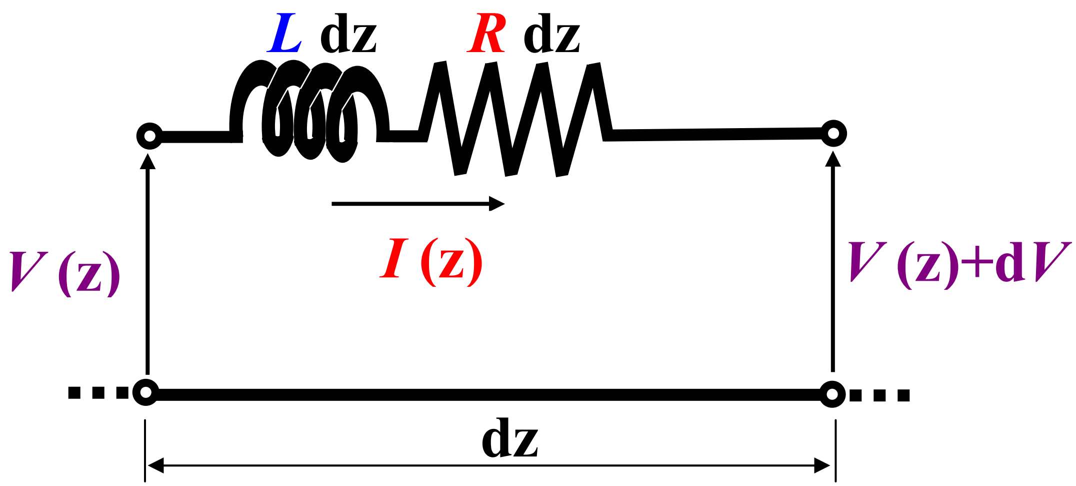
is the "characteristic impedance" of the loss-less transmission line.

❑ Lossy Transmission Line

The solution for a uniform lossy transmission line can be obtained with a very similar procedure, using the equivalent circuit for the elementary cell shown in the figure below.



The series impedance determines the variation of the voltage from input to output of the cell, according to the sub-circuit



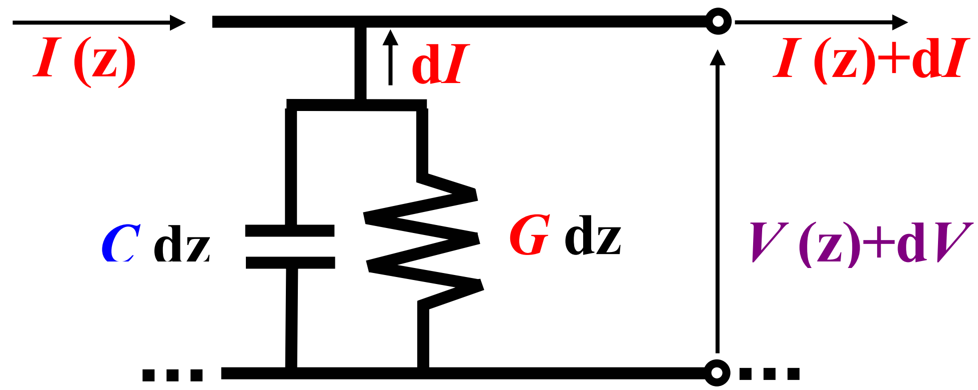
The corresponding circuit equation is

$$(V + \mathrm{d}V) - V = -(j\omega L\mathrm{d}z + R\mathrm{d}z)I$$

from which we obtain a first order differential equation for the voltage

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -(j\omega L + R)\, I$$

The current flowing through the shunt admittance determines the input-output variation of the current, according to the sub-circuit



The corresponding circuit equation is

$$\begin{aligned} \mathrm{d}I &= -(j\omega C\mathrm{dz} + G\mathrm{dz})(V + \mathrm{d}V) \\ &= -(j\omega C + G)V\mathrm{dz} - (j\omega C + G)\,\mathrm{d}V\,\mathrm{dz} \end{aligned}$$

The second term (including $\mathrm{d}V\ \mathrm{dz}$) can be ignored, giving a first order differential equation for the current

$$\frac{\mathrm{d}I}{\mathrm{dz}} = -(j\omega C + G)V$$

We have again a system of coupled first order differential equations that describe the behavior of voltage and current on the lossy transmission line

$$\begin{cases} \dfrac{\mathrm{d}V}{\mathrm{dz}} = -(j\omega L + R)\, I \\ \dfrac{\mathrm{d}I}{\mathrm{dz}} = -(j\omega C + G)\, V \end{cases}$$

These are the "telegraphers' equations" for the lossy transmission line case.

One can easily obtain a set of uncoupled equations by differentiating with respect to the coordinate z as done earlier

$$\boxed{\frac{dI}{dz} = -(j\omega C + G)V}$$

$$\boxed{\begin{aligned} \frac{d^2V}{dz^2} &= -(j\omega L + R)\frac{dI}{dz} = (j\omega L + R)(j\omega C + G)V \\ \frac{d^2I}{dz^2} &= -(j\omega C + G)\frac{dV}{dz} = (j\omega C + G)(j\omega L + R)I \end{aligned}}$$

$$\boxed{\frac{dV}{dz} = -(j\omega L + R)I}$$

These are the "telephonists' equations" for the lossy line.

The telephonists' equations for the lossy transmission line are uncoupled second order differential equations and are again wave equations. The general solution for the voltage equation is

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z} = V^+ e^{-\alpha z} e^{-j\beta z} + V^- e^{\alpha z} e^{j\beta z}$$

where the wave propagation constant is now the complex quantity

$$\gamma = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta$$

The real part α **of the propagation constant** γ **describes the attenuation of the signal due to resistive losses. The imaginary part** β **describes the propagation properties of the signal waves as in loss-less lines.**

The exponential terms including α **are "real", therefore, they only affect the "magnitude" of the voltage phasor. The exponential terms including** β **have unitary magnitude and purely "imaginary" argument, affecting only the "phase" of the waves in space.**

The current distribution on a lossy transmission line can be readily obtained by differentiation of the result for the voltage

$$\frac{\mathrm{d}V}{\mathrm{dz}} = -(j\omega L + R)\, I = -\gamma\, V^{+} e^{-\gamma z} + \gamma\, V^{-} e^{\gamma z}$$

which gives

$$I(z) = \sqrt{\frac{(j\omega C + G)}{(j\omega L + R)}}\, (V^{+} e^{-\gamma z} - V^{-} e^{\gamma z})$$

$$= \frac{1}{Z_0} (V^{+} e^{-\gamma z} - V^{-} e^{\gamma z})$$

with the "characteristic impedance" of the lossy transmission line

$$Z_0 = \sqrt{\frac{(j\omega L + R)}{(j\omega C + G)}}$$

Note: the characteristic impedance is now complex !

For both loss-less and lossy transmission lines

the characteristic impedance does not depend on the line length

but only on the metal of the conductors, the dielectric material surrounding the conductors and the geometry of the line cross-section, which determine L, R, C, and G.

One must be careful not to interpret the characteristic impedance as some lumped impedance that can replace the transmission line in an equivalent circuit.

This is a very common mistake!