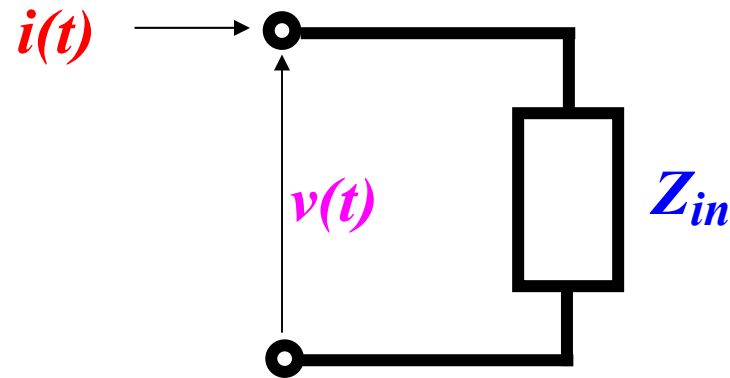


## Power in Circuits

Consider the input impedance of a transmission line circuit, with an applied voltage  $v(t)$  inducing an input current  $i(t)$ .



For sinusoidal excitation, we can write

$$v(t) = V_0 \cos(\omega t + \phi) \quad \phi \in [-\pi/2, \pi/2]$$

$$i(t) = I_0 \cos(\omega t)$$

where  $V_0$  and  $I_0$  are **peak values** and  $\phi$  is the **phase difference** between voltage and current. Note that  $\phi = 0$  only when the input impedance is real (purely resistive).

The **time-dependent** input **power** is given by

$$\begin{aligned} P(t) &= v(t) i(t) = V_0 I_0 \cos(\omega t + \phi) \cos(\omega t) \\ &= \frac{V_0 I_0}{2} [\cos(\phi) + \cos(2\omega t + \phi)] \end{aligned}$$

The power has two (**Fourier**) components:

**(A) an average value**

$$\frac{V_0 I_0}{2} \cos(\phi)$$

**(B) an oscillatory component with frequency  $2f$**

$$\frac{V_0 I_0}{2} \cos(2\omega t + \phi)$$

The power flow changes periodically in time with an oscillation like **(B)** about the average value **(A)**. Note that only when  $\phi = 0$  we have  $\cos(\phi) = 1$ , implying that for a resistive impedance the power is always **positive** (flowing from generator to load).

When **voltage** and **current** are out of phase, the average value of the power has lower magnitude than the peak value of the oscillatory component. Therefore, during portions of the period of oscillation the power can be **negative** (flowing from load to generator). This means that when the power flow is positive, the **reactive component** of the input impedance **stores energy**, which is reflected back to the generator side when the power flow becomes negative.

For an oscillatory excitation, we are interested in finding the behavior of the power during one full period, because from this we can easily obtain the **average** behavior in time. From the point of view of power consumption, we are also interested in knowing the power **dissipated** by the **resistive component** of the impedance.

Using  $\cos(A + B) = \cos A \cos B - \sin A \sin B$  one can write

$$v(t) = V_0 \cos(\omega t + \phi) = \underbrace{V_0 \cos \phi \cos \omega t}_{\text{in phase with current}} - \underbrace{V_0 \sin \phi \sin \omega t}_{\text{in quadrature with current}}$$

This gives an alternative expression for power:

$$\begin{aligned}
 P(t) &= V_0 \cos(\omega t) I_0 \cos(\omega t) \cos(\phi) - V_0 \cos(\omega t) I_0 \sin(\omega t) \sin(\phi) \\
 &= \underbrace{V_0 I_0 \cos(\phi) \cos^2(\omega t)}_{\text{Real Power}} - \underbrace{\frac{V_0 I_0}{2} \sin(\phi) \sin(2\omega t)}_{\text{Reactive Power}} \\
 &= \underbrace{\frac{V_0 I_0}{2} \cos(\phi) + \frac{V_0 I_0}{2} \cos(\phi) \cos(2\omega t)}_{\text{Real Power}} - \underbrace{\frac{V_0 I_0}{2} \sin(\phi) \sin(2\omega t)}_{\text{Reactive Power}}
 \end{aligned}$$

The **real power** corresponds to the power dissipated by the **resistive** component of the impedance, and it is always positive.

The **reactive power** corresponds to power stored and then reflected by the **reactive** component of the impedance. It oscillates from positive to negative during the period.

Until now we have discussed properties of instantaneous power. Since we are considering time-harmonic periodic signals, it is very convenient to consider the **time-average power**

$$\langle P(t) \rangle = \frac{1}{T} \int_0^T P(t) dt$$

where  $T = 1/f$  is the period of the oscillation.

To determine the time-average power, we can use either the Fourier or the real/reactive power formulation.

## Fourier representation

$$\begin{aligned}
 \langle P(t) \rangle &= \frac{1}{T} \int_0^T \frac{V_0 I_0}{2} \cos(\phi) dt + \frac{1}{T} \underbrace{\int_0^T \frac{V_0 I_0}{2} \cos(2\omega t - \phi) dt}_{=0} \\
 &= \frac{V_0 I_0}{2} \cos(\phi)
 \end{aligned}$$

As one should expect, the time-average power flow is simply given by the Fourier component corresponding to the average of the original signal.

## Real/Reactive power representation

$$\begin{aligned}
 \langle P(t) \rangle &= \frac{1}{2T} \left( \int_0^T V_0 I_0 \cos(\phi) dt + \underbrace{\int_0^T V_0 I_0 \cos(\phi) \cos(2\omega t) dt}_{=0} \right) \\
 &\quad - \frac{1}{2T} \underbrace{\int_0^T V_0 I_0 \sin(\phi) \sin(2\omega t) dt}_{=0} \\
 &= \frac{V_0 I_0}{2} \cos(\phi)
 \end{aligned}$$

This result tells us that the **time-average power** flow is the average of the **real power**. The **reactive power** has zero time-average, since power is stored and completely reflected by the reactive component of the input impedance during the period of oscillation.

The **maximum** of the **reactive power** is

$$\max \{ P_{reac} \} = \max \left\{ \frac{V_0 I_0}{2} \sin(\phi) \sin(2\omega t) \right\} = \frac{V_0 I_0}{2} \sin(\phi)$$

Since the time-average of the reactive power is zero, we often use the maximum value above as an indication of the reactive power.

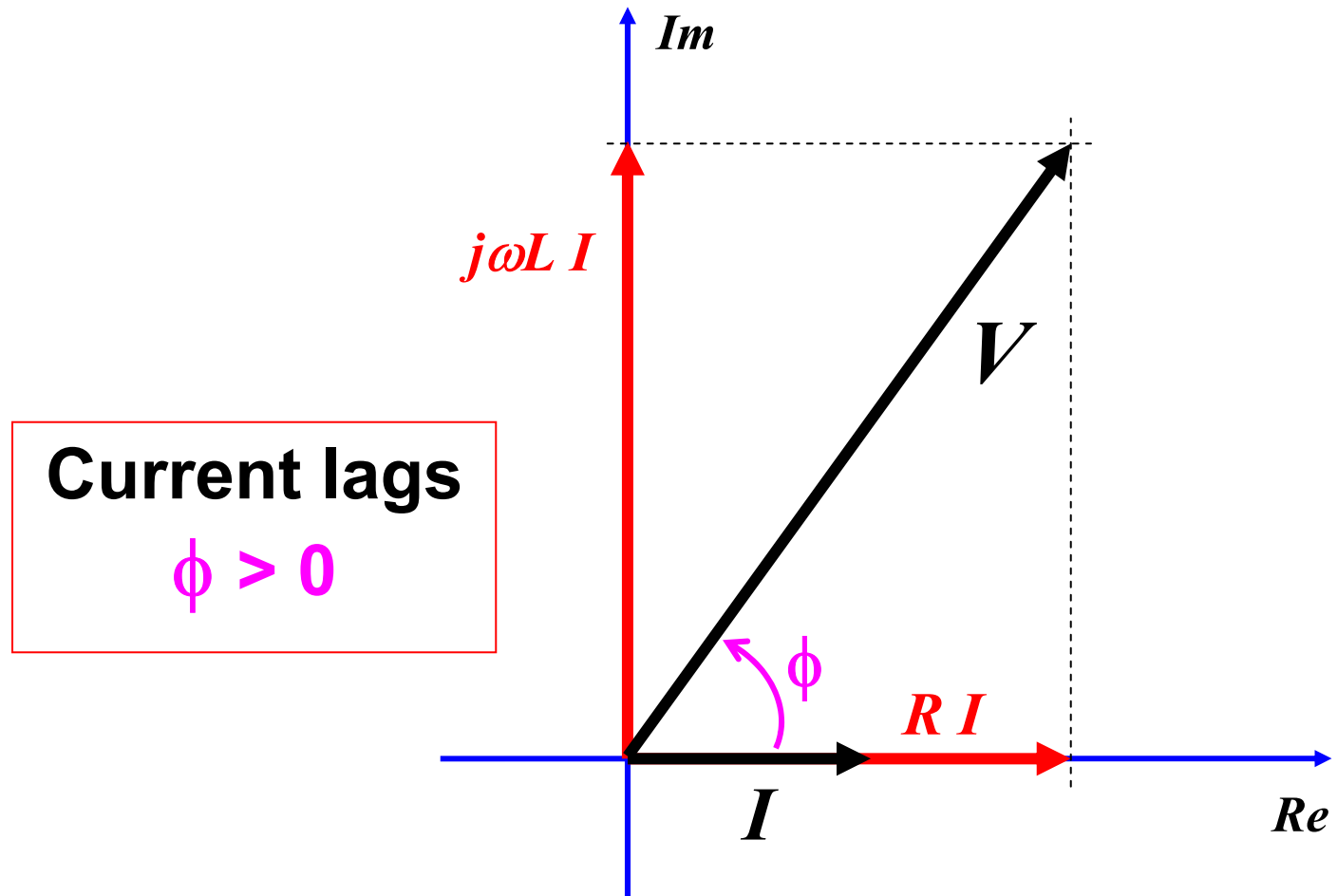
The sign of the phase  $\phi$  tells us about the **imaginary part of the impedance** or **reactance**:

- $\phi > 0$  **The reactance is inductive**  
**Current is lagging with respect to voltage**  
**Voltage is leading with respect to current**
- $\phi < 0$  **The reactance is capacitive**  
**Voltage is lagging with respect to current**  
**Current is leading with respect to voltage**



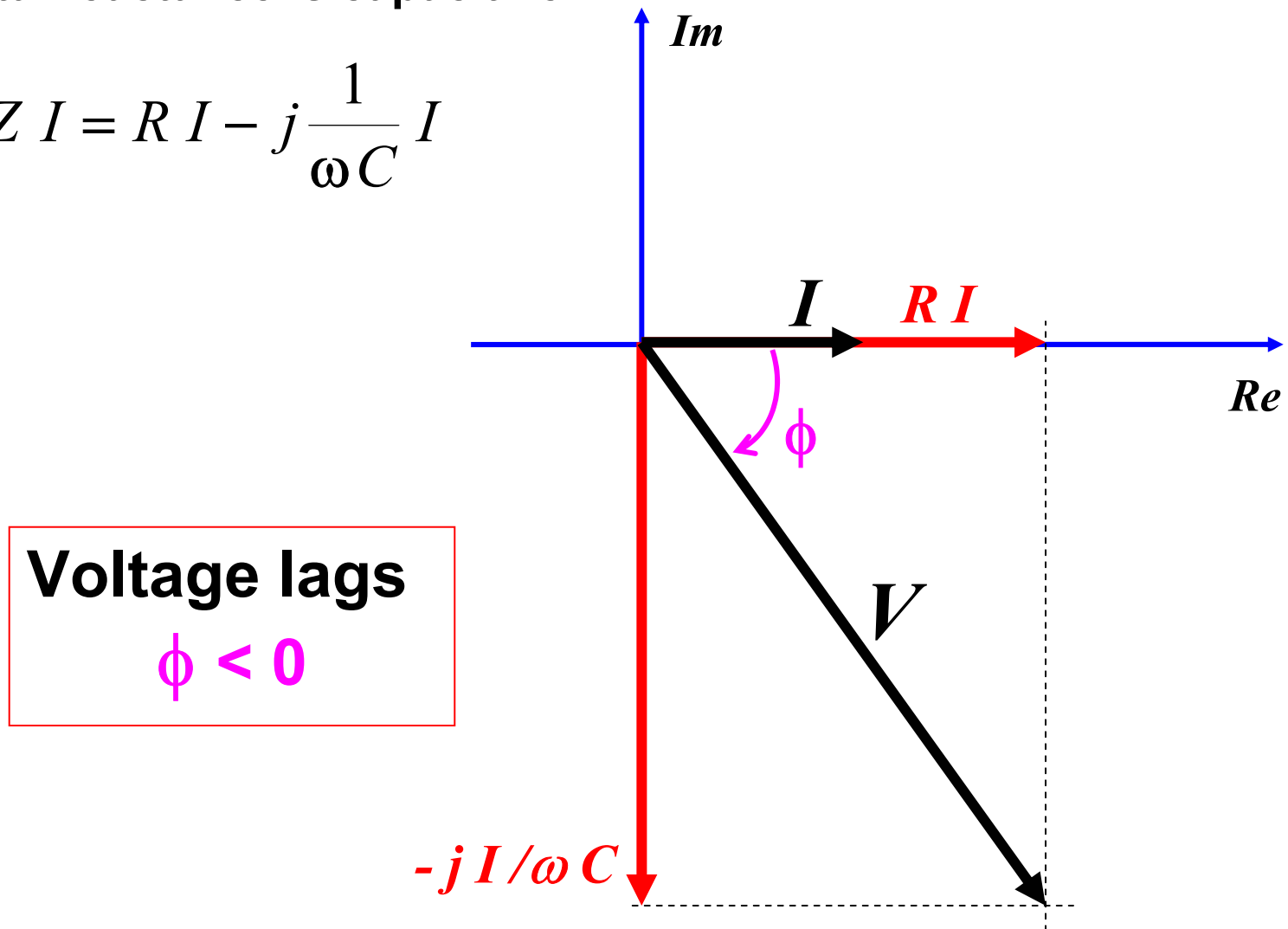
If the total reactance is inductive

$$V = Z I = R I + j\omega L I$$



If the total reactance is capacitive

$$V = Z I = R I - j \frac{1}{\omega C} I$$



In many situations, we may use the **root-mean-square** (r.m.s.) values of quantities, instead of the peak value. For a given signal

$$v(t) = V_0 \cos(\omega t)$$

the r.m.s. value is defined as

$$\begin{aligned} V_{rms} &= \sqrt{\frac{1}{T} \int_0^T V_0^2 \cos^2(\omega t) dt} = V_0 \sqrt{\frac{1}{\omega T} \int_0^{2\pi} \cos^2(\omega t) d\omega t} \\ &= V_0 \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \cos^2(\vartheta) d\vartheta} = \frac{1}{\sqrt{2}} V_0 \end{aligned}$$

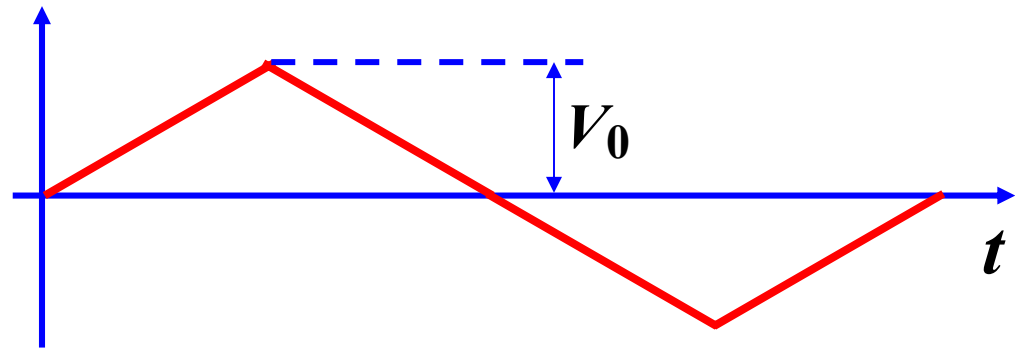
This result is valid for **sinusoidal signals**. Any given signal shape corresponds to a specific coefficient ( **peak factor** =  $V_0 / V_{rms}$  ) that allows one to convert directly from peak value to r.m.s. value.

The **peak factor** for **sinusoidal signals** is

$$\frac{V_0}{V_{rms}} = \sqrt{2} \approx 1.4142$$

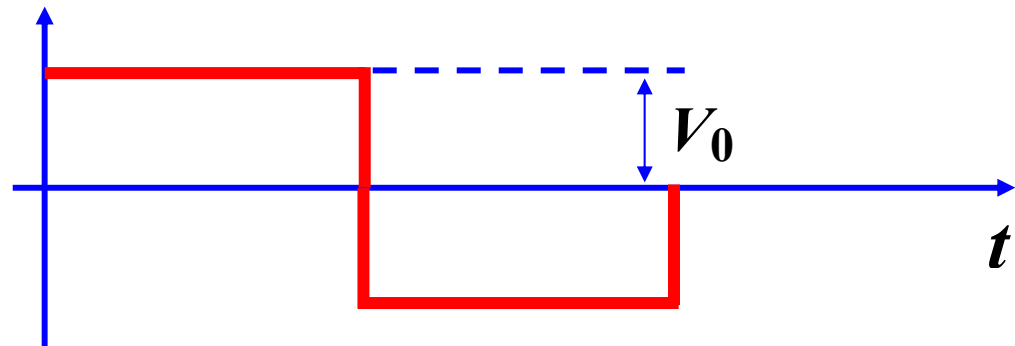
For a **symmetric triangular signal** the **peak factor** is

$$\frac{V_0}{V_{rms}} = \sqrt{3} \approx 1.732$$



For a **symmetric square signal** the **peak factor** is simply

$$\frac{V_0}{V_{rms}} = 1$$



For a **non-sinusoidal periodic signal**, we can use a decomposition into orthogonal **Fourier components** to obtain the r.m.s. value:

$$V(t) = V_{av} + V_1(t) + V_2(t) + V_3(t) \dots = \sum_k V_k(t)$$

$$\begin{aligned} [V_{rms}]^2 &= \frac{1}{T} \int_0^T V^2(t) dt = \frac{1}{T} \int_0^T [\sum_k V_k(t)]^2 dt = \\ &= \frac{1}{T} \int_0^T [\sum_k V_k^2(t)] dt + \frac{1}{T} \int_0^T [\sum_{i \neq j} 2 V_i(t) V_j(t)] dt = \\ &= \sum_k \left[ \frac{1}{T} \int_0^T V_k^2(t) dt \right] + \sum_{i \neq j} \left[ \frac{1}{T} \int_0^T \overbrace{2 V_i(t) V_j(t)}^{\text{orthogonal}} dt \right] = \sum_k (V_{k,rms})^2 \end{aligned}$$

$$\Rightarrow V_{rms} = \sqrt{\sum_k (V_{k,rms})^2} = \sqrt{V_{av}^2 + (V_{1,rms})^2 + (V_{2,rms})^2 + \dots}$$

The final result holds for any decomposition into orthogonal functions and it is known in mathematics as **Parseval's identity**.

In terms of **r.m.s. values**, the time-average power for a sinusoidal signal is then

$$\langle P(t) \rangle = \frac{V_0}{\sqrt{2}} \frac{I_0}{\sqrt{2}} \cos(\phi) = V_{rms} I_{rms} \cos(\phi)$$

Finally, we can relate the time-average power to the phasors of voltage and current. Since

$$v(t) = V_0 \cos(\omega t) = \text{Re} \{ V_0 \exp(j\omega t) \}$$

$$i(t) = I_0 \cos(\omega t - \phi) = \text{Re} \{ I_0 \exp(-j\phi) \exp(j\omega t) \}$$

we have phasors

$$V = V_0$$

$$I = I_0 \exp(-j\phi)$$

The time-average power in terms of phasors is given by

$$\begin{aligned}\langle P(t) \rangle &= \frac{1}{2} \operatorname{Re} \{ V I^* \} = \frac{1}{2} \operatorname{Re} \{ V_0 I_0 \exp(j\phi) \} \\ &= \frac{V_0 I_0}{2} \cos(\phi)\end{aligned}$$

Note that one must always use the **complex conjugate** of the **phasor current** to obtain the time-average power. It is important to remember this when voltage and current are expressed as functions of each other. Only when the impedance is purely resistive,  $I = I^* = I_0$  since  $\phi = 0$ .

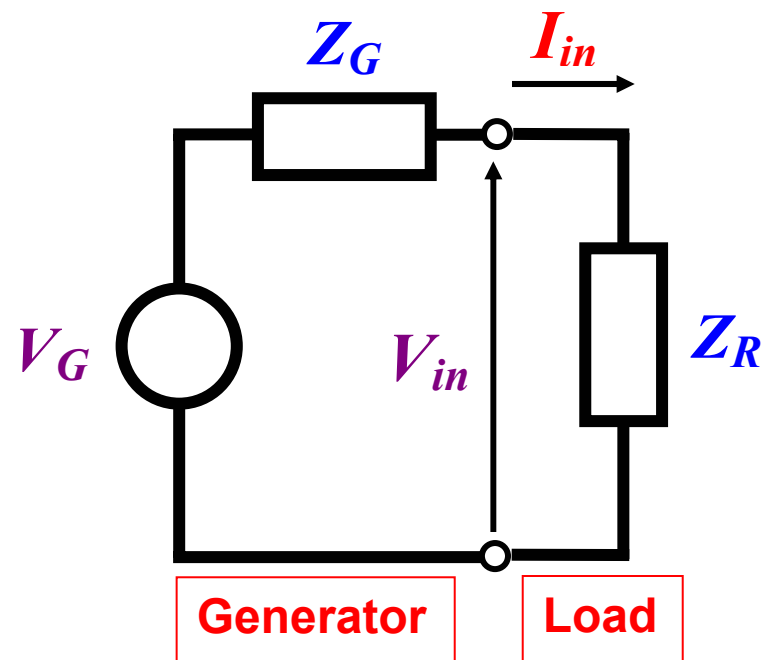
Also, note that the **time-average power** is always a **real positive** quantity and that it **is not** the phasor of the time-dependent power. It is a common mistake to think so.

Now we consider power flow including explicitly the generator, to understand in which conditions **maximum power transfer** to a load can take place.

$$V_{in} = V_G \frac{Z_R}{Z_G + Z_R}$$

$$I_{in} = V_G \frac{1}{Z_G + Z_R}$$

$$\langle P_{in} \rangle = \frac{1}{2} \text{Re} \{ V_{in} I_{in}^* \}$$





As a first case, we examine **resistive** impedances

$$Z_G = R_G \qquad Z_R = R_R$$

Voltage and current are **in phase** at the input. The time-average power dissipated by the load is

$$\begin{aligned} \langle P(t) \rangle &= \frac{1}{2} V_G \frac{R_R}{R_G + R_R} V_G^* \frac{1}{R_G + R_R} \\ &= \frac{1}{2} |V_G|^2 \frac{R_R}{(R_G + R_R)^2} \end{aligned}$$

To find the **load** resistance that **maximizes power transfer** to the load for a given generator we impose

$$\frac{d\langle P(t) \rangle}{dR_R} = 0$$

from which we obtain

$$\frac{d}{dR_R} \left[ \frac{R_R}{(R_G + R_R)^2} \right] = 0$$

$$\frac{(R_G + R_R)^2 - 2R_R(R_G + R_R)}{(R_G + R_R)^4} = 0$$

$$(R_G + R_R) - 2R_R = 0 \quad \Rightarrow \quad R_R = R_G$$

**We conclude that for maximum power transfer the load resistance must be identical to the generator resistance.**

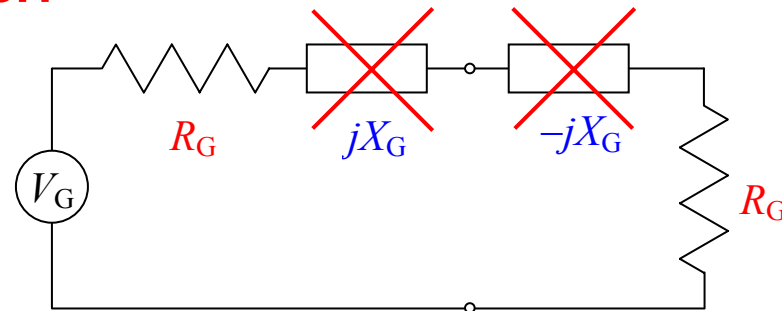
Let's consider now **complex impedances**

$$Z_R = R_R + jX_R$$

$$Z_G = R_G + jX_G$$

**For maximum power transfer, generator and load impedances must be complex conjugate of each other:**

$$Z_R = Z_G^* \Rightarrow \begin{aligned} R_R &= R_G \\ X_R &= -X_G \end{aligned}$$



This can be easily understood by considering that, **to maximize** the active **power** supplied to the load, **voltage** and **current** of the generator should remain **in phase**. If the reactances of generator and load are opposite and cancel each other along the path of the current, the generator will only see a resistance. Voltage and current will be in phase with maximum power delivered to the load.

The **total time-average power** supplied by the **generator** in conditions of maximum power transfer is

$$\langle P_{tot} \rangle = \frac{1}{2} \operatorname{Re} \{ V_G I_{in}^* \} = \frac{1}{2} |V_G|^2 \frac{1}{2R_R} = \frac{1}{4} |V_G|^2 \frac{1}{R_R}$$

The **time-average power** supplied to the **load** is

$$\begin{aligned} \langle P_{in} \rangle &= \frac{1}{2} \operatorname{Re} \{ V_{in} I_{in}^* \} = \frac{1}{2} \operatorname{Re} \left\{ V_G \frac{Z_R}{Z_G + Z_R} V_G^* \left( \frac{1}{Z_G + Z_R} \right)^* \right\} \\ &= \frac{1}{2} |V_G|^2 \operatorname{Re} \left\{ \frac{R_R + jX_R}{4R_R^2} \right\} = \frac{1}{8} |V_G|^2 \frac{1}{R_R} \end{aligned}$$

The **power dissipated** by the **internal generator impedance** is

$$\begin{aligned}\langle P_G \rangle &= \frac{1}{2} \operatorname{Re} \{ (V_G - V_{in}) I_{in}^* \} \\ &= \frac{1}{4} |V_G|^2 \frac{1}{R_R} - \frac{1}{8} |V_G|^2 \frac{1}{R_R} = \frac{1}{8} |V_G|^2 \frac{1}{R_R}\end{aligned}$$

We conclude that, in conditions of **maximum power transfer**, only **half** of the **total active power** supplied by the **generator** is actually used by the **load**. The generator impedance dissipates the remaining half of the available active power.

**This may seem a disappointing result, but it is the best one can do for a real generator with a given internal impedance!**