Complex Numbers, Phasors and Circuits

Complex numbers are defined by points or vectors in the complex plane, and can be represented in Cartesian coordinates

$$z = a + jb \qquad\qquad j = \sqrt{-1}$$

or in polar (exponential) form

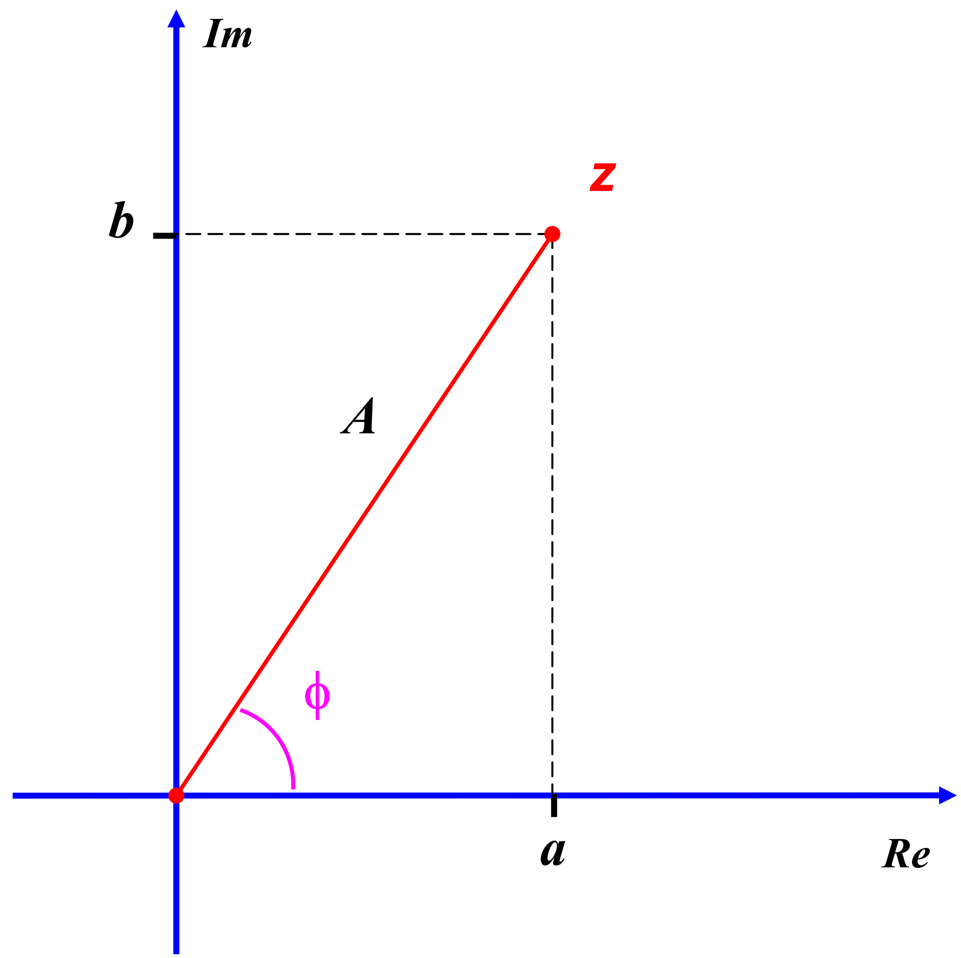
$$z = A\exp(j\phi) = A\cos(\phi) + jA\sin(\phi)$$

$$a = A\cos(\phi)$$ **real part**

$$b = A\sin(\phi)$$ **imaginary part**

where

$$A = \sqrt{a^2 + b^2} \qquad\qquad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$



Note : $z = A\exp(j\phi) = A\exp(j\phi \pm j2n\pi)$

Every complex number has a complex conjugate

$$z^* = \left(a + jb\right)^* = a - jb$$

so that

$$z \cdot z^* = (a + jb) \cdot (a - jb)$$
$$= a^2 + b^2 = |\, z \,|^2 = A^2$$

In polar form we have

$$z^* = \left(A \exp(j\phi)\right)^* = A \exp(-j\phi)$$
$$= A \exp\left(j2\pi - j\phi\right)$$
$$= A \cos\left(\phi\right) - jA \sin\left(\phi\right)$$

The polar form is more useful in some cases. For instance, when raising a complex number to a power, the Cartesian form

$$z^n = (a + jb) \cdot (a + jb) \ldots (a + jb)$$

is cumbersome, and impractical for non–integer exponents. In polar form, instead, the result is immediate

$$z^n = \left[A \exp(j\phi) \right]^n = A^n \exp\left(jn\phi \right)$$

In the case of roots, one should remember to consider $\phi + 2k\pi$ as argument of the exponential, with k = integer, otherwise possible roots are skipped:

$$\sqrt[n]{z} = \sqrt[n]{A \exp\left(j\phi + j2k\pi \right)} = \sqrt[n]{A} \exp\left(j\frac{\phi}{n} + j\frac{2k\pi}{n} \right)$$

The results corresponding to angles up to 2π are solutions of the root operation.

In electromagnetic problems it is often convenient to keep in mind the following simple identities

$$j = \exp\left(j\frac{\pi}{2} \right) \qquad\qquad - j = \exp\left(- j\frac{\pi}{2} \right)$$

It is also useful to remember the following expressions for trigonometric functions

$$\cos\left(z \right) = \frac{\exp(jz) + \exp(-jz)}{2} \quad ; \quad \sin\left(z \right) = \frac{\exp(jz) - \exp(-jz)}{2\,j}$$

resulting from Euler's identity

$$\exp(\pm jz) = \cos(z) \pm j\sin(z)$$

Complex representation is very useful for time-harmonic functions of the form

$$A\cos(\omega t+\phi)=\mathrm{Re}\left[A\exp(j\omega t+j\phi)\right]$$
$$=\mathrm{Re}\left[A\exp(j\phi)\exp(j\omega t)\right]$$
$$=\mathrm{Re}\left[\bar{A}\exp(j\omega t)\right]$$

The complex quantity

$$\bar{A}=A\exp(j\phi)$$

contains all the information about amplitude and phase of the signal and is called the phasor of

$$A\cos(\omega t+\phi)$$

If it is known that the signal is time-harmonic with frequency ω, the phasor completely characterizes its behavior.

Often, a time-harmonic signal may be of the form:

$$A \sin(\omega t + \phi)$$

and we have the following complex representation

$$\begin{aligned} A \sin(\omega t + \phi) &= \mathrm{Re}\left[-jA\left(\cos(\omega t + \phi) + j \sin(\omega t + \phi)\right)\right] \\ &= \mathrm{Re}\left[-jA \exp(j\omega t + j\phi)\right] \\ &= \mathrm{Re}\left[A \exp(-j\pi/2) \exp(j\phi) \exp(j\omega t)\right] \\ &= \mathrm{Re}\left[A \exp\left(j(\phi - \pi/2)\right) \exp(j\omega t)\right] \\ &= \mathrm{Re}\left[\bar{A} \exp(j\omega t)\right] \end{aligned}$$

with phasor

$$\bar{A} = A \exp\left(j(\phi - \pi/2)\right)$$

This result is not surprising, since

$$\cos(\omega t + \phi - \pi/2) = \sin(\omega t + \phi)$$

Time differentiation can be greatly simplified by the use of phasors. Consider for instance the signal

$$V(t) = V_0 \cos(\omega t + \phi) \qquad \text{with phasor} \qquad \bar{V} = V_0 \exp(j\phi)$$

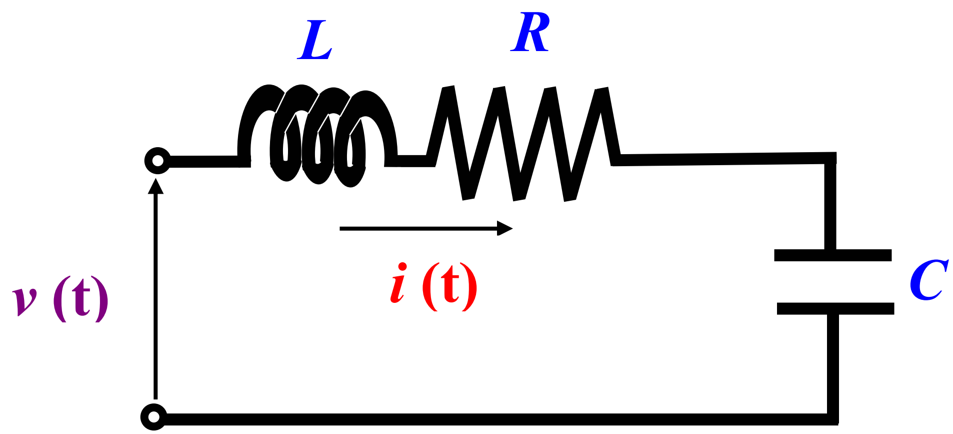
The time derivative can be expressed as

$$\frac{\partial V(t)}{\partial t} = -\omega V_0 \sin(\omega t + \phi)$$

$$= \mathrm{Re}\{ j\omega V_0 \exp(j\phi) \exp(j\omega t)\}$$

$$\Rightarrow \quad j\omega V_0 \exp(j\phi) = j\omega \bar{V} \qquad \text{is the phasor of} \qquad \frac{\partial V(t)}{\partial t}$$

With phasors, time-differential equations for time harmonic signals can be transformed into algebraic equations. Consider the simple circuit below, realized with lumped elements



This circuit is described by the integro-differential equation

$$v(t) = L\frac{d\,i(t)}{dt} + R\,i + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

Upon time-differentiation we can eliminate the integral as

$$\frac{d\,v(t)}{dt} = L\frac{d^2\,i(t)}{dt^2} + R\frac{d\,i}{dt} + \frac{1}{C}\;i(t)$$

If we assume a time-harmonic excitation, we know that voltage and current should have the form

$$v(t) = V_0\cos(\omega t + \alpha_V) \qquad \textbf{phasor} \Rightarrow \qquad V = V_0\exp(j\alpha_V)$$

$$i(t) = I_0\cos(\omega t + \alpha_I) \qquad \textbf{phasor} \Rightarrow \qquad I = I_0\exp(j\alpha_I)$$

If V_0 and α_V are given,

$\Rightarrow$ **I_0 and α_I are the unknowns of the problem.**

The differential equation can be rewritten using phasors

$$L\,\mathrm{Re}\left\{-\omega^2 I \exp(j\omega t)\right\} + R\,\mathrm{Re}\left\{j\omega I \exp(j\omega t)\right\}$$

$$+\frac{1}{C}\mathrm{Re}\left\{I \exp(j\omega t)\right\} = \mathrm{Re}\left\{j\omega V \exp(j\omega t)\right\}$$

Finally, the transform phasor equation is obtained as

$$V = \left(R + j\omega L - j\frac{1}{\omega C}\right) I = Z\,I$$

where

$$\underbrace{Z}_{\text{Impedance}} = \underbrace{R}_{\text{Resistance}} + j\underbrace{\left(\omega L - \frac{1}{\omega C}\right)}_{\text{Reactance}}$$

The result for the phasor current is simply obtained as

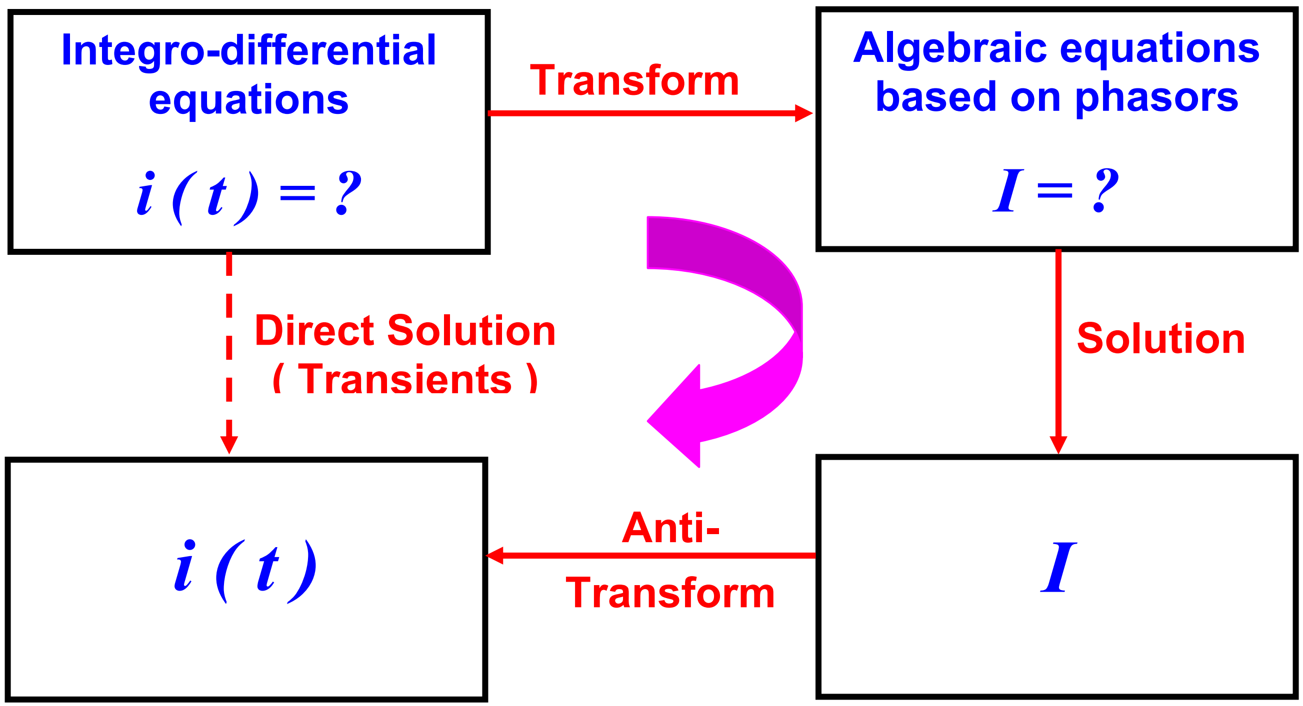
$$I = \frac{V}{Z} = \frac{V}{\left(R + j\omega L - j\dfrac{1}{\omega C} \right)} = I_0 \exp\left(j\alpha_I \right)$$

which readily yields the unknowns I_0 and α_I .

The time dependent current is then obtained from

$$\begin{aligned} i(t) &= \mathrm{Re}\left\{ I_0 \exp\left(j\alpha_I \right) \exp\left(j\omega t \right) \right\} \\ &= I_0 \cos\left(\omega t + \alpha_I \right) \end{aligned}$$

The phasor formalism provides a convenient way to solve time-harmonic problems in steady state, without having to solve directly a differential equation. The key to the success of phasors is that with the exponential representation one can immediately separate frequency and phase information. Direct solution of the time-dependent differential equation is only necessary for transients.



The phasor representation of the circuit example above has introduced the concept of impedance. Note that the resistance is not explicitly a function of frequency. The reactance components are instead linear functions of frequency:

Inductive component $\Rightarrow$ proportional to ω

Capacitive component $\Rightarrow$ inversely proportional to ω

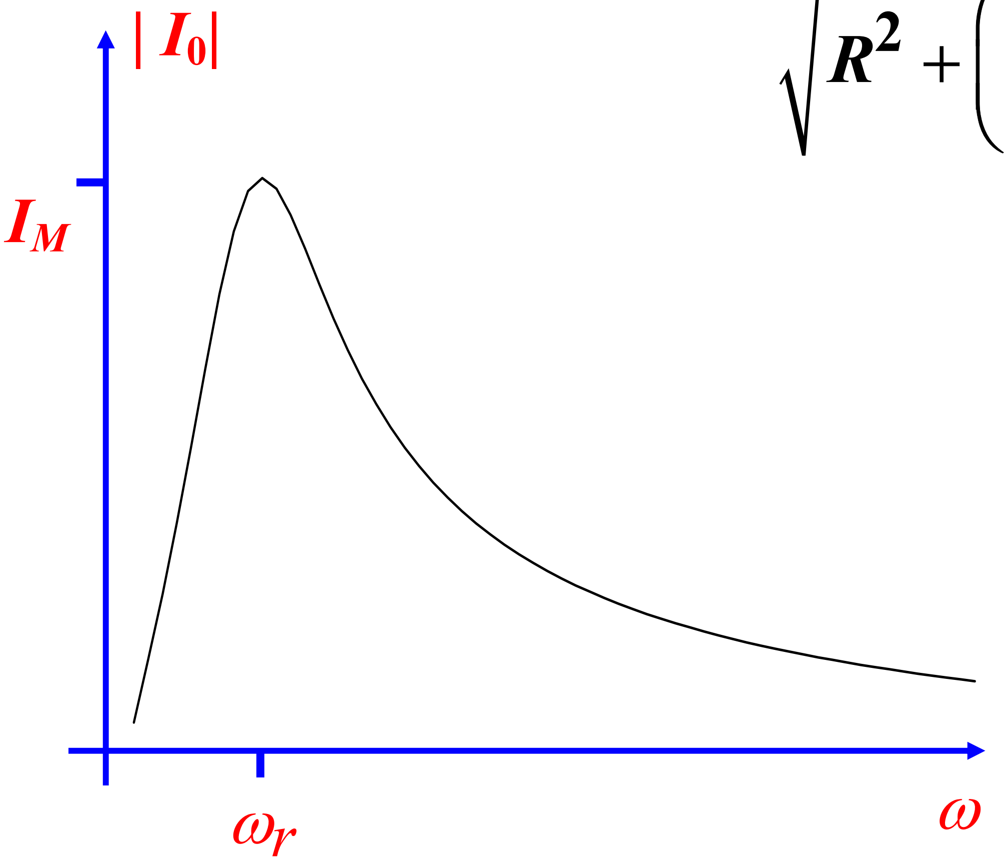
Because of this frequency dependence, for specified values of L and C, one can always find a frequency at which the magnitudes of the inductive and capacitive terms are equal

$$\omega_r L = \frac{1}{\omega_r C} \qquad \Rightarrow \qquad \omega_r = \frac{1}{\sqrt{LC}}$$

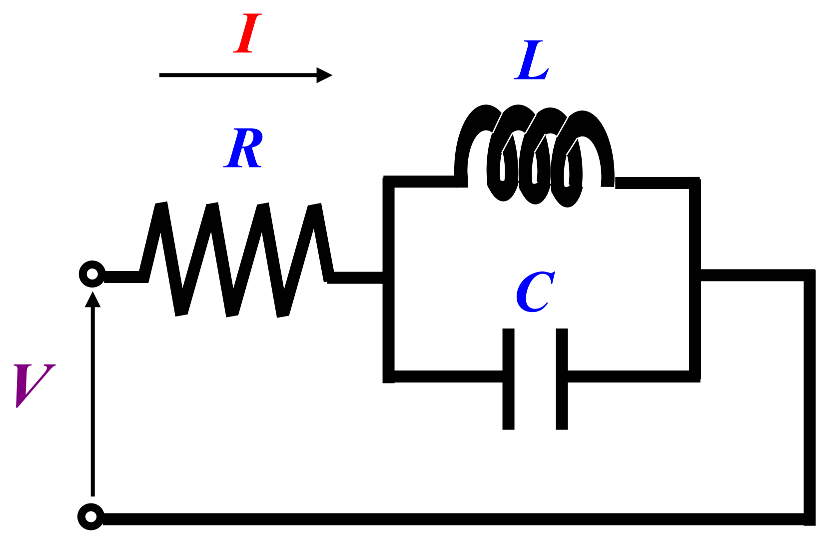
This is a resonance condition. The reactance cancels out and the impedance becomes purely resistive.

The peak value of the current phasor is maximum at resonance

$$\left|I_0\right| = \frac{\left|V_0\right|}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$



Consider now the circuit below where an inductor and a capacitor are in parallel



The input impedance of the circuit is

$$Z_{in} = R + \left(\frac{1}{j\omega L} + j\omega C \right)^{-1} = R + \frac{j\omega L}{1 - \omega^2 LC}$$

When

$$\omega = 0 \qquad Z_{in} = R$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad Z_{in} \to \infty$$

$$\omega \to \infty \qquad Z_{in} = R$$

At the resonance condition

$$\omega_r = \frac{1}{\sqrt{LC}}$$

the part of the circuit containing the reactance components behaves like an open circuit, and no current can flow. The voltage at the terminals of the parallel circuit is the same as the input voltage V.